Coupling matter to spinfoam models using Higher Gauge Theory

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INTRODUCTION

A short recap of the spinfoam quantization method:

- Step 1: Rewrite the GR action — as a topological BF theory plus simplicity constraint,

$$S_{\text{Plebanski}}[B,\omega,\phi] = \int_{\mathcal{M}_4} \langle B \wedge F(\omega)\rangle_{\mathfrak{g}} + \langle \phi(B \wedge B)\rangle_{\mathfrak{g}} \,,$$

  where the Lie group G is Lorentz-like, and $\mathfrak{g}$ is its Lie algebra.

- Step 2: Quantize the topological sector — a state sum over a triangulated manifold $T(\mathcal{M}_4)$,

$$Z_{BF} = \sum_{\Lambda} \quad \prod_{v} \mathcal{A}_v(\Lambda) \prod_{e} \mathcal{A}_e(\Lambda) \prod_{\Delta} \mathcal{A}_{\Delta}(\Lambda) \prod_{\tau} \mathcal{A}_{\tau}(\Lambda) \prod_{\sigma} \mathcal{A}_{\sigma}(\Lambda) \,.$$

  "Colors" Λ are reps of G, amplitudes $\mathcal{A}$ chosen so that Z_{BF} is invariant wrt. Pachner moves.

- Step 3: Impose the simplicity constraint — deform the invariant Z_{BF} by modifying the amplitudes and reps,

$$Z_{BF} \to Z_{GR} : \qquad \mathcal{A}(\Lambda) \to W(j) \,, \qquad j = f(\Lambda) \,,$$

  obtaining a state sum Z_{GR} which defines a spinfoam model (Barret-Crane, EPRL/FK, etc).

Key question — how to add matter fields into the above?

HIGHER CATEGORY THEORY

A flash introduction to higher category theory:

- An n-category is a set of *objects* with:
  - *morphisms* (maps between objects),
  - 2-*morphisms* (maps between morphisms),
  - 3-*morphisms* (maps between 2-morphisms), ... up to n-*morphisms*,

  along with certain axioms to provide suitable rules for composition, associativity, etc.
- An n-group is a special case of an n-category, which has only one object, and all morphisms are invertible.

A more detailed introduction to higher category theory:

⇒ look up **"An Invitation to Higher Gauge Theory"** [Baez, Huerta (2011)]

The purpose of n-groups (for physicists):

⇒ *more fine-grained description of symmetry* using an n-group, than using a group,

⇒ generalization of differential geometry: *parallel transport, connection, holonomy, curvature.*

LIE 3-GROUPS

Focus on a Lie 3-group, specified in detail by a 2-crossed module:

[Faria Martins, Picken (2011); Wang (2014)]

$$(\quad L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G \quad , \quad \triangleright \quad , \quad \{\_\,,\_\} \quad)$$

- L, H, G — Lie groups,
- δ, ∂ — boundary morphisms,
- $\triangleright$ — action of G, $\quad \triangleright : G \times G \to G, \quad \triangleright : G \times H \to H, \quad \triangleright : G \times L \to L,$
- $\{\_\,,\_\}$ — Peiffer lifting, $\quad \{\_\,,\_\} : H \times H \to L.$

Axioms that hold among these maps:

Chain complex:	$\partial\delta = 1_G,$
Conjugation:	$g \triangleright g_0 = g\, g_0\, g^{-1},$
G-equivariance of ∂ and δ:	$g \triangleright \partial h = \partial(g \triangleright h)\,, \quad g \triangleright \delta l = \delta(g \triangleright l),$
G-equivariance of lifting:	$g \triangleright \{h_1, h_2\} = \{g \triangleright h_1, g \triangleright h_2\},$
Peiffer commutator:	$\delta\,\{h_1, h_2\} = h_1 h_2 h_1^{-1} (\partial h_1) \triangleright h_2^{-1},$
L-commutator:	$\{\delta l_1, \delta l_2\} = l_1 l_2 l_1^{-1} l_2^{-1},$
δ-lifting relation:	$\{\delta l, h\}\,\{h, \delta l\} = l(\partial h \triangleright l^{-1}),$
Left product rule:	$\{h_1 h_2, h_3\} = \{h_1, h_2 h_3 h_2^{-1}\}\,\partial h_1 \triangleright \{h_2, h_3\}.$

LIE 3-GROUPS

Purpose of all this — to generalize the notion of parallel transport, from curves to surfaces to volumes:

- Connection generalized to a 3-connection (α, β, γ), a triple of algebra-valued differential forms:

$$\begin{array}{rclll} \alpha & = & \alpha^{\alpha}{}_{\mu}(x) & \tau_{\alpha} \otimes \mathbf{d}x^{\mu} & \in \mathfrak{g} \otimes \Lambda^1(\mathcal{M})\,, \\ \beta & = & \frac{1}{2}\,\beta^{a}{}_{\mu\nu}(x) & t_a \otimes \mathbf{d}x^{\mu} \wedge \mathbf{d}x^{\nu} & \in \mathfrak{h} \otimes \Lambda^2(\mathcal{M})\,, \\ \gamma & = & \frac{1}{3!}\,\gamma^{A}{}_{\mu\nu\rho}(x) & T_A \otimes \mathbf{d}x^{\mu} \wedge \mathbf{d}x^{\nu} \wedge \mathbf{d}x^{\rho} & \in \mathfrak{l} \otimes \Lambda^3(\mathcal{M})\,. \end{array}$$

- Line holonomy generalized to surface and volume holonomies:

$$g = \mathcal{P}\exp \int_{\mathcal{P}_1} \alpha\,, \qquad h = \mathcal{S}\exp \int_{\mathcal{S}_2} \beta\,, \qquad l = \mathcal{V}\exp \int_{\mathcal{V}_3} \gamma\,.$$

- Ordinary curvature generalized to 3-curvature $(\mathcal{F}, \mathcal{G}, \mathcal{H})$, where:

$$\begin{array}{rcl} \mathcal{F} & = & \mathbf{d}\alpha + \alpha \wedge \alpha - \partial\beta\,, \\ \mathcal{G} & = & \mathbf{d}\beta + \alpha \wedge^{\triangleright} \beta - \delta\gamma\,, \\ \mathcal{H} & = & \mathbf{d}\gamma + \alpha \wedge^{\triangleright} \gamma - \{\beta \wedge \beta\}\,. \end{array}$$

HIGHER GAUGE THEORY

At this point one can construct the action for a higher gauge theory:

$$S_{3BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}} \,.$$

$\Rightarrow$ Topological $3BF$ theory, based on the 3-group $(\, L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G \,,\, \triangleright \,,\, \{\_\,,\_\} \,)$.

The physical interpretation of the Lagrange multipliers C and D:

- for $H = \mathbb{R}^4$, multiplier C can be ***interpreted as the tetrad* 1*-form***:

$$C \quad \to \quad e \quad = \quad e^a{}_\mu(x)\, t_a \otimes \mathbf{d}x^\mu \,,$$

[Miković, MV (2012)]

- for given L, multiplier D can be ***interpreted as the set of matter fields***:

$$D \quad \to \quad \phi \quad = \quad \phi^A(x)\, T_A \,.$$

[Radenković, MV (2019)]

$\Rightarrow$ The action thus becomes:

$$S_{3BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle e \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle \phi \wedge \mathcal{H} \rangle_{\mathfrak{l}} \,.$$

THE STANDARD MODEL

How many real-valued field components do we have in the Standard Model?

The fermion sector gives us:

$$\left.\begin{array}{cccc} \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L & \begin{pmatrix} u_r \\ d_r \end{pmatrix}_L & \begin{pmatrix} u_g \\ d_g \end{pmatrix}_L & \begin{pmatrix} u_b \\ d_b \end{pmatrix}_L \\ \\ (\nu_e)_R & (u_r)_R & (u_g)_R & (u_b)_R \\ \\ (e^-)_R & (d_r)_R & (d_g)_R & (d_b)_R \end{array}\right\} = 16 \ \frac{\text{spinors}}{\text{family}} \times$$

$$\times 3 \text{ families } \times 4 \ \frac{\text{real-valued components}}{\text{spinor}} \ = 192 \text{ real-valued components } \phi^A \,.$$

The Higgs sector gives us:

$$\left.\begin{pmatrix} \phi^+ \\ \phi_0 \end{pmatrix}\right\} = 2 \text{ complex scalar fields } = 4 \text{ real-valued components } \phi^A \,.$$

This suggests the structure for L in the form:

$$L = \mathbb{R}^4(\mathbb{C}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G}) \,,$$

where $\mathbb{G}$ is the Grassmann algebra.

THE STANDARD MODEL

The actions $\triangleright : G \times L \to L$ and $\triangleright : G \times H \to H$ specify the transformation properties of matter ϕ^A and tetrad $e^a{}_\mu$ with respect to Lorentz and internal symmetries:

- Choose the group $G = SO(3,1) \times SU(3) \times SU(2) \times U(1)$. Then, for example, given any $g \in G$ and a doublet
$$\begin{pmatrix} u_b \\ d_b \end{pmatrix}_L ,$$
the action $g \triangleright u_b$ encodes that u_b consists of 4 real-valued fields which transform as:
  - a left-handed spinor wrt. $SO(3,1)$,
  - as a "blue" component of the fundamental representation of $SU(3)$,
  - and as "isospin $+\frac{1}{2}$" of the left doublet wrt. $SU(2) \times U(1)$.
- Next choose the group $H = \mathbb{R}^4$. The action $\triangleright$ of G on H is via vector representation for the $SO(3,1)$ part and via trivial representation for the $SU(3) \times SU(2) \times U(1)$ part.
- Finally, the other maps in the 3-group are chosen to be trivial. For all $l \in L$ and $\vec{u}, \vec{v} \in H$,
$$\delta l = 1_H = 0 \,, \qquad \partial \vec{v} = 1_G \,, \qquad \{\vec{u}, \vec{v}\} = 1_L \,.$$

THE STANDARD MODEL

The Standard Model* 3*-group, $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G \,,\, \rhd \,,\, \{\_\,,\_\})$***, defined as:***

$$G = SO(3,1)\times SU(3)\times SU(2)\times U(1)\,, \qquad H = \mathbb{R}^4\,,$$

$$L = \mathbb{R}^4(\mathbb{C})\times \mathbb{R}^{64}(\mathbb{G})\times \mathbb{R}^{64}(\mathbb{G})\times \mathbb{R}^{64}(\mathbb{G})\,.$$

The constrained $3BF$ ***action for the Standard Model coupled to Einstein-Cartan gravity:***

$$S_{GR+SM} = \int_{\mathcal{M}_4} B_{\hat\alpha}\wedge \mathcal{F}^{\hat\alpha} + e_{\hat a}\wedge \mathcal{G}^{\hat a} + \phi_{\hat A}\wedge \mathcal{H}^{\hat A}$$

$$+\Big(B_{\hat\alpha} - C_{\hat\alpha}{}^{\hat\beta} M_{cd\hat\beta} e^c\wedge e^d\Big)\wedge\lambda^{\hat\alpha} - \Big(\gamma_{\hat A} - e^a\wedge e^b\wedge e^c C_{\hat A}{}^{\hat B} M_{abc\hat B}\Big)\wedge\lambda^{\hat A} - 4\pi i\, l_p^2\,\varepsilon_{abcd} e^a\wedge e^b\wedge\beta^c\phi_{\hat A} T^{d\hat A}{}_{\hat B}\phi^{\hat B}$$

$$+\zeta^{ab}{}_{\hat\alpha}\wedge\Big(M_{ab}{}^{\hat\alpha}\varepsilon^{cdef} e_c\wedge e_d\wedge e_e\wedge e_f - F^{\hat\alpha}\wedge e_c\wedge e_d\Big) + \zeta^{ab}{}_{\hat A}\wedge\Big(M_{abc}{}^{\hat A}\varepsilon^{cdef} e_d\wedge e_e\wedge e_f - F^{\hat A}\wedge e_a\wedge e_b\Big)$$

$$-\varepsilon_{abcd} e^a\wedge e^b\wedge e^c\wedge e^d\ \Big(\Lambda + M_{\hat A\hat B}\phi^{\hat A}\phi^{\hat B} + Y_{\hat A\hat B\hat C}\phi^{\hat A}\phi^{\hat B}\phi^{\hat C} + L_{\hat A\hat B\hat C\hat D}\phi^{\hat A}\phi^{\hat B}\phi^{\hat C}\phi^{\hat D}\Big)\,.$$

⇒ Finally, one can go even further and separate scalar and fermion fields into distinct groups, employing the structure of a 4-group and a $4BF$ action. [Miković, MV (2021)]

QUANTIZATION

Revisit the spinfoam quantization method:

- Step 1: Rewrite the GR+SM action... **done!**
- Step 2: Quantize the topological sector... **done!** (almost) [Radenković, MV (2022)]

$$
\begin{aligned}
Z \;=\; & |G|^{-|v|+|e|-|\Delta|} \; |H|^{|v|-|e|+|\Delta|-|\tau|} \; |L|^{-|v|+|e|-|\Delta|+|\tau|-|\sigma|} \\
& \prod_{e\in T} \int_G dg_e \;\prod_{\Delta\in T} \int_H dh_\Delta \;\prod_{\tau\in T} \int_L dl_\tau \\
& \prod_{\Delta\in T} \delta_G(\; \partial(h_\Delta) g_1 g_2 g_3^{-1} \;) \prod_{\tau\in T} \delta_H(\; \delta(l_\tau) h_1 (g \triangleright h_2) h_3^{-1} h_4^{-1} \;) \\
& \prod_{\sigma\in T} \delta_L(\; l_1^{-1} h_1 \triangleright' \{h_2, (g_1 g_2) \triangleright h_3\} l_2^{-1} (h_4 \triangleright' l_3) l_4 h_5 \triangleright' (g_3 \triangleright l_5) \;)
\end{aligned}
$$

  Invariant wrt. $4D$ **Pachner moves!**

- Step 3: Impose the simplicity constraints... **work in progress!**

QUANTIZATION

GR without matter can be described using* 2*-groups* $(H \stackrel{\partial}{\to} G, \triangleright)$*:

- Topological $2BF$ theory developed and studied: [Girelli, Pfeiffer, Popescu (2008)] [Miković, Martins (2011)]

$$S_{2BF} = \int_{\mathcal{M}_4} B^{ab} \wedge \mathcal{F}_{ab} + C^a \wedge \mathcal{G}_a \,.$$

- The choice $G = SO(3,1)$, $H = \mathbb{R}^4$, is called the *Poincaré* 2*-group*. The action for GR is

$$S_{GR} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab}(\omega) + e^a \wedge G_a - \phi_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} \, e_c \wedge e_d \right) \,.$$

  One possible quantization prescription leads to the *spincube model.* [Miković, MV (2012)]

- Representation theory for 2-groups (including the Poincaré 2-group), has been developed in great detail. [Baez, Baratin, Freidel, Wise (2012)]

- The topological invariant and TQFT for the *Euclidean* 2*-group* ($G = SO(4)$, $H = \mathbb{R}^4$) has also been studied in detail. [Baratin, Freidel (2015)] [Asante, Dittrich, Girelli, Riello, Tsimiklis (2020)]

CONCLUSIONS

- Higher gauge theory represents a formalism where gravity, gauge fields, fermions and Higgs are treated on an equal footing.
- Resulting generalized spinfoam models naturally include matter fields coupled to gravity.
- The underlying algebraic structure of a 3-group classifies all fundamental fields by specifying groups L, H, G and their maps $\delta, \partial, \triangleright, \{\_\,,\_\}$.
- This structure has natural geometrical interpretation of parallel transport along a curve, a surface, and a volume.
- The gauge group L specifies the complete matter sector of the Standard Model if one chooses

$$L = \mathbb{R}^4(\mathbb{C}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G})\,.$$

- The action $\triangleright$ of G on L specifies the transformation properties of matter fields.
- Nontrivial choices of the 3-group structure may provide new avenues for research on unification of all fields.

THANK YOU!