RELATION BETWEEN L_∞ ALGEBRAS AND HIGHER CATEGORY THEORY

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TOPICS

- **Introduction**
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- **Differential crossed modules**
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INTRODUCTION

The main statement in brief:

Lie algebra	$\longleftrightarrow$	Lie group
L_n algebra	$\longleftrightarrow$	Lie n-group
L_∞ algebra	$\longleftrightarrow$	Lie ∞-group

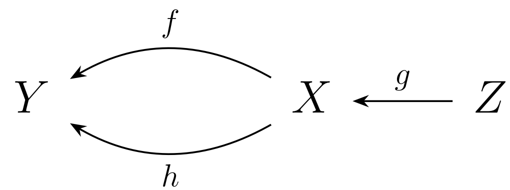
For physics, the interesting cases are $n = 1, 2, 3, 4, 6, \infty$.

HIGHER CATEGORY THEORY

A flash introduction to the category theory "ladder":

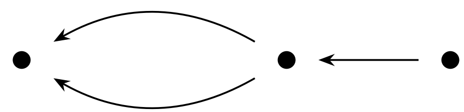
- a 0-category $\mathcal{C}_0 = Obj$ is just a set. The elements of the set are called *objects*.
- a 1-category $\mathcal{C}_1 = (Obj, Mor)$ is a structure which has objects and morphisms between them,

$$X, Y, Z, \dots \in Obj\,, \qquad f, g, h, \dots \in Mor\,,$$



such that certain rules are respected, like the associativity of morphism composition, etc.

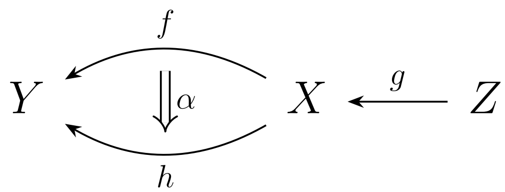
The most famous example is the category whose objects are dots on the paper, and morphisms are arrows connecting the dots:



HIGHER CATEGORY THEORY

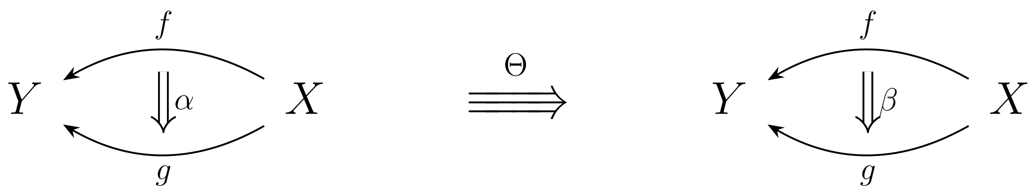
- a 2-category $\mathcal{C}_2 = (Obj, Mor_1, Mor_2)$ is a structure which has objects, morphisms between them, and morphisms between morphisms, called 2-morphisms,

$$X, Y, Z, \dots \in Obj\,, \qquad f, g, h, \dots \in Mor_1\,, \qquad \alpha, \beta, \dots \in Mor_2\,,$$



such that similar rules as above are respected.

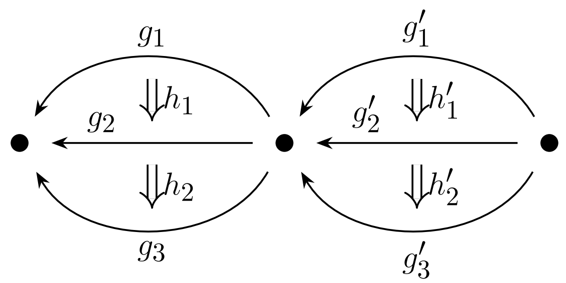
- a 3-category $\mathcal{C}_3 = (Obj, Mor_1, Mor_2, Mor_3)$ additionally has morphisms between 2-morphisms, called 3-morphisms, again with a certain set of axioms about compositions.



n-GROUPS AND CROSSED MODULES

The algebraic structure of a group is a special case of a category:

- a group is a category with only one object, while all morphisms are invertible,
- a 2-group is a 2-category with only one object, while all 1-morphisms and 2-morphisms are invertible,
- a 3-group is a 3-category with only one object, while all 1-morphisms, 2-morphisms and 3-morphisms are invertible,
- and similarly for a general n-group ($n \in \mathbb{N}$ and $n = \infty$).



n-GROUPS AND CROSSED MODULES

A more practical way to talk about a 2-group — a crossed module:

$$(H \stackrel{\partial}{\to} G \, , \, \triangleright)$$

- G and H are ordinary groups,
- ∂ is called a "boundary" homomorphism, $\partial : H \to G$,
- $\triangleright$ is called an "action" of G onto both groups G and H,

$$\triangleright : G \times G \to G \, , \qquad \triangleright : G \times H \to H \, ,$$

- and certain axioms are assumed to hold,

$$\begin{array}{lrcl} \textbf{conjugation:} & g \triangleright g' & = & g g' g^{-1} \, , \\ \partial\textbf{-} \triangleright \ \textbf{relation:} & \partial(g \triangleright h) & = & g \triangleright \partial h \, , \\ \textbf{Peiffer identity:} & \partial h \triangleright h' & = & h h' h^{-1} \, , \end{array}$$

for all $g, g' \in G$ and $h, h' \in H$.

n-GROUPS AND CROSSED MODULES

A more practical way to talk about a 3-group — a 2-crossed module:

$$(\, L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G \, , \, \triangleright \, , \, \{\_ \, , \_\} \,)$$

- G, H and L are ordinary groups,
- there are two boundary homomorphisms, δ and ∂,

$$\delta : L \to H \, , \qquad \partial : H \to G \, ,$$

- there is a defined action $\triangleright$ of G onto all three groups G, H and L,

$$\triangleright : G \times G \to G \, , \qquad \triangleright : G \times H \to H \, , \qquad \triangleright : G \times L \to L \, ,$$

- there is a bracket operation called *Peiffer lifting* over H to L,

$$\{\_ \, , \_\} : H \times H \to L \, ,$$

- and certain axioms are assumed to hold among all these maps.

n-GROUPS AND CROSSED MODULES

The axioms of a 2-crossed module $L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G$:

chain complex:	$\partial\delta l$	$=$	$1_G\,,$
conjugation:	$g \triangleright g'$	$=$	$gg'g^{-1}\,,$
∂- $\triangleright$ relation:	$\partial(g \triangleright h)$	$=$	$g \triangleright \partial h\,,$
δ- $\triangleright$ relation:	$\delta(g \triangleright l)$	$=$	$g \triangleright \delta l\,,$
Peiffer commutator:	$\delta\,\{h,h'\}$	$=$	$hh'h^{-1}(\partial h \triangleright h')^{-1}\,,$
G-equivariance of lifting:	$g \triangleright \{h,h'\}$	$=$	$\{g \triangleright h, g \triangleright h'\}\,,$
commutator in L:	$\{\delta l, \delta l'\}$	$=$	$l\,l'\,l^{-1}\,l'^{-1}\,,$
δ-lifting relation:	$l(\partial h \triangleright l)^{-1}$	$=$	$\{\delta l, h\}\,\{h, \delta l\}\,,$
left product:	$\{hh', \tilde{h}\}$	$=$	$\{h, h'\tilde{h}h'^{-1}\}\,\partial h \triangleright \{h', \tilde{h}\}\,,$
right product:	$\{h, h'\tilde{h}\}$	$=$	$\{h,h'\}\,\{h,\tilde{h}\}\,\{(\partial h \triangleright h')hh'^{-1}h^{-1}, \partial h \triangleright h'\}\,,$

for all $g \in G$, $h \in H$ and $l \in L$. . .

DIFFERENTIAL CROSSED MODULES

Why focus attention on *Lie* n-groups?

- In addition to being groups, Lie groups are also *manifolds*:

$$\textit{algebra} \quad \longleftrightarrow \quad \textit{geometry}$$

- Being a manifold, a Lie group has a **tangent space**. Being a group, a Lie group induces a **product** between tangent vectors — a **Lie algebra**.
- Moreover, given a Lie algebra, one can **reconstruct** a Lie group (though not uniquely), providing a **correspondence** between Lie algebras and Lie groups.
- Can a similar correspondence be obtained more generally, for Lie n-groups? **Yes!**

$$\begin{array}{ccc} \text{Lie } n\text{-group} & \Longleftrightarrow & \text{Lie } (n-1)\text{-crossed module} \\ & & \updownarrow \\ L_n \text{ algebra} & \Longleftrightarrow & \text{differential } (n-1)\text{-crossed module} \end{array}$$

DIFFERENTIAL CROSSED MODULES

Given a Lie crossed module $(\; H \stackrel{\partial}{\to} G \;,\; \triangleright \;)$**, one can introduce a *differential crossed module*:**

$$(\; \mathfrak{h} \stackrel{\partial}{\to} \mathfrak{g} \;,\; \triangleright \;)$$

- $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras of Lie groups G and H,
- boundary homomorphism ∂ and the action $\triangleright$ are induced:

$$\partial : \mathfrak{h} \to \mathfrak{g} \,, \qquad \triangleright : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \,, \qquad \triangleright : \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h} \,,$$

- and the corresponding linearized axioms hold:

$$\begin{array}{lrcl} \textbf{adjoint action:} & \underline{g} \triangleright \underline{g}' & = & [\underline{g}, \underline{g}'] \,, \\ \partial\textbf{-} \triangleright \; \textbf{relation:} & \partial(\underline{g} \triangleright \underline{h}) & = & \underline{g} \triangleright \partial \underline{h} \,, \\ \textbf{Peiffer identity:} & \partial \underline{h} \triangleright \underline{h}' & = & [\underline{h}, \underline{h}'] \,, \end{array}$$

for all $\underline{g}, \underline{g}' \in \mathfrak{g}$ and $\underline{h}, \underline{h}' \in \mathfrak{h}$.

DIFFERENTIAL CROSSED MODULES

Similarly, given a Lie 2**-crossed module** $(\, L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G \;,\; \triangleright \;,\; \{\_\,,\_\} \,)$**, one can introduce a** ***differential*** 2***-crossed module*****:**

$$(\, \mathfrak{l} \stackrel{\delta}{\to} \mathfrak{h} \stackrel{\partial}{\to} \mathfrak{g} \;,\; \triangleright \;,\; \{\_\,,\_\} \,)$$

- where $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{l}$ are Lie algebras of G, H, L,
- all maps δ, ∂, $\triangleright$ and $\{\_\,,\_\}$ are inherited from the 3-group by linearization,

$$\delta : \mathfrak{l} \to \mathfrak{h} \,, \qquad \partial : \mathfrak{h} \to \mathfrak{g} \,,$$

$$\triangleright : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \,, \qquad \triangleright : \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h} \,, \qquad \triangleright : \mathfrak{g} \times \mathfrak{l} \to \mathfrak{l} \,,$$
$$\{\_\,,\_\} : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{l} \,,$$

- corresponding linearized axioms apply.

L_∞ AND L_n ALGEBRAS

Let us now recall the definition of the L_n algebra. It consists of:

- an n-graded vector space,

$$V = \bigoplus_{k=0}^{n-1} V_k \, ,$$

- a set of multilinear *brackets* $l_k(\_,\dots,\_)$ of k arguments, for every $k \in \mathbb{N}$,

$$l_k : \underbrace{V \otimes \cdots \otimes V}_{k \text{ terms}} \to V \, ,$$

- which are totally graded-antisymmetric,

$$l_k(\dots, v, v', \dots) = -(-1)^{|v||v'|} l_k(\dots, v', v, \dots) \, ,$$

- and satisfy homotopy relations

$$J_k(\{v\}) \equiv \sum_{i=1}^{k} (-1)^{i(k-i)} \sum_{\sigma} \chi(\sigma, \{v\}) \; l_{k+1-i}(l_i(v_{\sigma(1)}, \dots, v_{\sigma(i)}), v_{\sigma(i+1)}, \dots, v_{\sigma(k)}) = 0 \, .$$

ISOMORPHISM

Finally, we are ready to establish an isomorphism — given a differential Lie crossed module $(\,\mathfrak{h} \stackrel{\partial}{\to} \mathfrak{g}\,,\,\triangleright\,)$, define the L_2 algebra as follows:

- since Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are vector spaces, define the graded vector space as

$$V_0 = \mathfrak{g}\,, \qquad V_1 = \mathfrak{h}\,, \qquad V \equiv V_0 \oplus V_1 = \mathfrak{g} \oplus \mathfrak{h}\,,$$

- for all $\underline{g} \in \mathfrak{g}$ and $\underline{h} \in \mathfrak{h}$, introduce the unary bracket $l_1(\_)$ as

$$l_1(\underline{g}) = 0\,, \qquad l_1(\underline{h}) = \partial \underline{h}\,,$$

- similarly introduce the binary bracket $l_2(\_\,,\_)$ as

$$l_2(\underline{g}, \underline{g}') = \underline{g} \triangleright \underline{g}'\,, \qquad l_2(\underline{g}, \underline{h}) = \underline{g} \triangleright \underline{h}\,, \qquad l_2(\underline{h}, \underline{h}') = 0\,,$$

- for $k \geqslant 3$, define all remaining brackets as

$$l_k(\_\,, \dots, \_) = 0\,.$$

ISOMORPHISM

Given these definitions, all axioms of the differential crossed module are equivalent to the nontrivial parts of the graded antisymmetry and homotopy relations for the L_2 algebra.

For example, the homotopy relation $J_2(v,v')=0$ states

$$l_1(l_2(v,v')) = l_2(l_1(v),v') + (-1)^{|v||v'|} l_2(v, l_1(v'))\,.$$

Evaluating this relation for $v=\underline{g}$, $v'=\underline{h}$, we have $|\underline{g}|=0$, $|\underline{h}|=1$, and using the definitions of brackets $l_1(\_)$ and $l_2(\_\,,\_)$, one obtains

$$\partial(\underline{g}\triangleright\underline{h}) = 0 + \underline{g}\triangleright\partial\underline{h}\,.$$

This is nothing but the axiom called ∂-$\triangleright$ **relation** of the differential crossed module.

In a similar manner, using the homotopy relations, one can recover all axioms of the differential crossed module. Conversely, starting from the axioms of the differential crossed module and the previous definition of L_2 algebra, one can prove that all homotopy relations hold.

ISOMORPHISM

In a similar fashion, one can establish another isomorphism — given a differential Lie 2-crossed module $(\, \mathfrak{l} \xrightarrow{\delta} \mathfrak{h} \xrightarrow{\partial} \mathfrak{g} \,,\, \triangleright \,,\, \{\_\,,\_\} \,)$**, define the** L_3 **algebra as follows:**

- since Lie algebras $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$ are vector spaces, define the graded vector space as

$$V_0 = \mathfrak{g}\,, \qquad V_1 = \mathfrak{h}\,, \qquad V_2 = \mathfrak{l}\,, \qquad V \equiv V_0 \oplus V_1 \oplus V_2 = \mathfrak{g} \oplus \mathfrak{h} \oplus \mathfrak{l}\,,$$

- for all $\underline{g} \in \mathfrak{g}$, $\underline{h} \in \mathfrak{h}$, and $\underline{l} \in \mathfrak{l}$ introduce the unary bracket $l_1(\_)$ as

$$l_1(\underline{g}) = 0\,, \qquad l_1(\underline{h}) = \partial \underline{h}\,, \qquad l_1(\underline{l}) = \delta \underline{l}\,,$$

- introduce the binary bracket $l_2(\_\,,\_)$ as

$$l_2(\underline{g},\underline{g}') = \underline{g} \triangleright \underline{g}'\,, \qquad l_2(\underline{g},\underline{h}) = \underline{g} \triangleright \underline{h}\,, \qquad l_2(\underline{g},\underline{l}) = \underline{g} \triangleright \underline{l}\,,$$

$$l_2(\underline{h},\underline{h}') = -\{\underline{h},\underline{h}'\} - \{\underline{h}',\underline{h}\}\,, \qquad l_2(\underline{h},\underline{l}) = -\{\delta\underline{l},\underline{h}\}\,, \qquad l_2(\underline{l},\underline{l}') = 0,$$

- for $k \geqslant 3$, define all remaining brackets as $l_k(\_\,,\dots,\_) = 0$.

ISOMORPHISM

Again, given these definitions, one can demonstrate the equivalence between all axioms of the differential 2-crossed module and the homotopy relations for the L_3 algebra.

As a simplest example, the homotopy relation $J_1(v) = 0$ states that

$$l_1(l_1(v)) = 0\,.$$

Evaluating this relation for $v = \underline{l}$ and using the definitions of the bracket $l_1(\_)$ one obtains

$$\partial\delta\underline{l} = 0\,,$$

which is the *chain complex* axiom of the differential 2-crossed module.

These two examples illustrate the stated relationship between Lie n-groups and L_n algebras.

PHYSICS AND HIGHER GAUGE THEORY

The main purpose of all these higher structures is to generalize the notion of parallel transport from curves to surfaces to volumes etc.:

- The differential Lie 2-crossed module $(\,\mathfrak{l} \xrightarrow{\delta} \mathfrak{h} \xrightarrow{\partial} \mathfrak{g}\,,\ \triangleright\,,\ \{\_\,,\_\}\,)$ enables one to define a 3-connection (α, β, γ) as a triple of 3-algebra-valued differential forms,

$$
\begin{aligned}
\alpha &= \alpha^{\alpha}{}_{\mu}(x)\, \tau_{\alpha}\, \mathbf{d}x^{\mu} && \in \Lambda^1(\mathcal{M}, \mathfrak{g})\,, \\
\beta &= \tfrac{1}{2}\beta^{a}{}_{\mu\nu}(x)\, t_a\, \mathbf{d}x^{\mu} \wedge \mathbf{d}x^{\nu} && \in \Lambda^2(\mathcal{M}, \mathfrak{h})\,, \\
\gamma &= \tfrac{1}{3!}\gamma^{A}{}_{\mu\nu\rho}(x)\, T_A\, \mathbf{d}x^{\mu} \wedge \mathbf{d}x^{\nu} \wedge \mathbf{d}x^{\rho} && \in \Lambda^3(\mathcal{M}, \mathfrak{l})\,.
\end{aligned}
$$

- Then introduce the line, surface and volume holonomies,

$$
g = \mathcal{P}\mathrm{exp} \int_{\mathcal{P}_1} \alpha\,, \qquad h = \mathcal{S}\mathrm{exp} \int_{\mathcal{S}_2} \beta\,, \qquad l = \mathcal{V}\mathrm{exp} \int_{\mathcal{V}_3} \gamma\,,
$$

- and corresponding curvature forms,

$$
\begin{aligned}
\mathcal{F} &= \mathbf{d}\alpha + \alpha \wedge \alpha - \partial\beta\,, \\
\mathcal{G} &= \mathbf{d}\beta + \alpha \wedge^{\triangleright} \beta - \delta\gamma\,, \\
\mathcal{H} &= \mathbf{d}\gamma + \alpha \wedge^{\triangleright} \gamma - \{\beta \wedge \beta\}\,.
\end{aligned}
$$

PHYSICS AND HIGHER GAUGE THEORY

At this point one can construct the so-called $3BF$ theory, with the action:

$$S_{3BF} = \int_{\mathcal{M}} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}} \,.$$

Here $\langle \_\,, \_ \rangle_{\mathfrak{g}}$, $\langle \_\,, \_ \rangle_{\mathfrak{h}}$, and $\langle \_\,, \_ \rangle_{\mathfrak{l}}$ represent the $\mathfrak{g}$-invariant nondegenerate symmetric bilinear forms over the Lie algebras $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{l}$. In terms of the L_3 algebra, these are components of the single such form defined over the graded vector space $V = \mathfrak{g} \oplus \mathfrak{h} \oplus \mathfrak{l}$,

$$\langle \_\,, \_ \rangle : V \otimes V \to \mathbb{R} \,.$$

The $3BF$ action is an example of a topological higher gauge theory, whose symmetry is described by the corresponding 3-group.

Non-topological actions (which are more interesting for physics) can be obtained from the $3BF$ action by adding appropriate simplicity constraint terms into the action, which impose appropriate equations of motion. There also exists a generalization to $4BF$ action, based on a 4-group structure, which has some additional benefits.

PHYSICS AND HIGHER GAUGE THEORY

Physical interpretation:

- all gauge fields are described by the $\mathfrak{g}$-valued 1-form part of the 3-connection,

$$\alpha = \alpha^{\alpha}{}_{\mu}(x)\,\tau_{\alpha}\,\mathbf{d}x^{\mu} \qquad \in \Lambda^{1}(\mathcal{M},\mathfrak{g})\,,$$

  which includes all internal gauge fields, as well as the Lorentz spin connection;

- the $\mathfrak{h}$-valued Lagrange multiplier 1-form C can be interpreted as the tetrad field,

$$C \to e = e^{a}{}_{\mu}(x)\,t_{a}\,\mathbf{d}x^{\mu} \qquad \in \Lambda^{1}(\mathcal{M},\mathfrak{h})\,,$$

  which (together with the spin connection) describes the gravitational field;

- the $\mathfrak{l}$-valued Lagrange multiplier 0-form D can be interpreted as the set of matter fields:

$$D \to \phi = \phi^{A}(x)\,T_{A} \qquad \in \Lambda^{0}(\mathcal{M},\mathfrak{l})\,,$$

  which include all fermions and scalar fields;

- in the $4BF$ theory, based on a 4-group, fermions and scalar fields are further separated into two different Lagrange multipliers.

REFERENCES

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THANK YOU!