Quantum gravity and elementary particles from higher gauge theory

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TOPICS

- **Introduction**
- **Category theory and 3-groups**
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- **Higher gauge theories**
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INTRODUCTION

The line of development that leads to higher gauge theories (HGT):

- Covariant LQG framework → "path integral on a lattice" idea

  → rewrite GR as a BF theory with a simplicity constraint (Plebanski action)

  → lattice quantization scheme → *spinfoam models*.

- Drawback — BF action has no fundamental tetrads (only classically emergent)
  → hard to couple matter fields to spinfoams!

  Solution — categorical generalization from BF to $2BF$ (from *group* to *2-group*)

  → tetrads become fundamental → straightforward to couple matter to gravity!

  First example of a HGT — *spincube model*. Nevertheless, matter is still ad hoc!

- Another categorical generalization — from $2BF$ to $3BF$ (*2-group* to *3-group*)

  → combine *all fields* (gravity, gauge, matter) into a unified algebraic structure.

  ⇒ ***Categorical unification*** — potential to explain fermion families, etc.

CATEGORY THEORY AND 3-GROUPS

A flash introduction to the category theory "ladder":

- a category $\mathcal{C} = (Obj, Mor)$ is a structure which has objects and morphisms between them,

$$X, Y, Z, \dots \in Obj\,, \qquad f, g, h, \dots \in Mor\,,$$

$$f : X \to Y, \quad g : Z \to X, \quad h : X \to Y, \dots$$

such that certain rules are respected, like the associativity of morphism composition, etc.

- a 2-category $\mathcal{C}_2 = (Obj, Mor_1, Mor_2)$ is a structure which has objects, morphisms between them, and morphisms between morphisms, called 2-morphisms,

$$X, Y, Z, \dots \in Obj\,, \qquad f, g, h, \dots \in Mor_1\,, \qquad \alpha, \beta, \dots \in Mor_2\,,$$

$$f : X \to Y, \quad g : Z \to X, \quad h : X \to Y, \dots \qquad \alpha : f \to h\,, \dots$$

such that similar rules as above are respected.

CATEGORY THEORY AND 3-GROUPS

- a 3-category $\mathcal{C}_3 = (Obj, Mor_1, Mor_2, Mor_3)$ additionally has morphisms between 2-morphisms, called 3-morphisms,

$$\Theta, \Phi, \dots \in Mor_3\,, \qquad \Theta : \alpha \to \beta\,, \dots$$

  again with a certain set of axioms about compositions of various n-morphisms.

- one can further generalize these structures to introduce 4-categories, n-categories, ∞-categories, etc.

The algebraic structure of a group is a special case of a category:

- a group is a category with only one object, while all morphisms are invertible;
- a 2-group is a 2-category with only one object, while all 1-morphisms and 2-morphisms are invertible;
- a 3-group is a 3-category with only one object, while all 1-morphisms, 2-morphisms and 3-morphisms are invertible.

LIE 3-GROUPS

A more practical way to talk about 3-group — a 2-crossed module:

$$L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G \,,$$

- for our purposes L, H and G are ordinary Lie groups,
- there are two "boundary homomorphisms" δ and ∂,
- there is a defined action $\triangleright$ of G onto G, H and L,

$$\triangleright : G \times G \to G \,, \qquad \triangleright : G \times H \to H \,, \qquad \triangleright : G \times L \to L \,,$$

- there is a bracket operation (*Peiffer lifting*) over H to L,

$$\{ \quad , \quad \} : H \times H \to L \,,$$

- and certain axioms are assumed to hold true among all these maps.

LIE 3-GROUPS

The axioms of a 2-crossed module $L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G$:

$$g \rhd \partial h = \partial(g \rhd h)\,,$$

$$g \rhd \delta l = \delta(g \rhd l)\,,$$

$$g \rhd g_0 = g\, g_0\, g^{-1}\,,$$

$$g \rhd \{h_1, h_2\} = \{g \rhd h_1, g \rhd h_2\}\,,$$

$$\partial\delta = 1_G\,,$$

$$\delta\,\{h_1, h_2\} = h_1 h_2 h_1^{-1} (\partial h_1) \rhd h_2^{-1}\,,$$

$$\{\delta l_1, \delta l_2\} = l_1 l_2 l_1^{-1} l_2^{-1}\,,$$

$$\{h_1 h_2, h_3\} = \{h_1, h_2 h_3 h_2^{-1}\}\,\partial h_1 \rhd \{h_2, h_3\}\,,$$

$$\{\delta l, h\}\,\{h, \delta l\} = l(\partial h \rhd l^{-1})\,.$$

. . . for all $g \in G$, $h \in H$ and $l \in L$. . . :-)

LIE 3-GROUPS

A Lie 3-group has a corresponding Lie 3-algebra, i.e. a differential 2-crossed module:

$$\mathfrak{l} \xrightarrow{\delta} \mathfrak{h} \xrightarrow{\partial} \mathfrak{g}\,,$$

- where $\mathfrak{l}$, $\mathfrak{h}$, $\mathfrak{g}$ are Lie algebras of L, H, G,
- the maps δ, ∂, $\triangleright$ and $\{\ ,\ \}$ are inherited from the 3-group,
- "corresponding" axioms apply.

In addition to all this, Lie algebras have their own Lie structure:

- generators,

$$T_A \in \mathfrak{l}\,, \qquad t_a \in \mathfrak{h}\,, \qquad \tau_\alpha \in \mathfrak{g}$$

- structure constants,

$$[T_A, T_B] = f_{AB}{}^C T_C\,, \qquad [t_a, t_b] = f_{ab}{}^c t_c\,, \qquad [\tau_\alpha, \tau_\beta] = f_{\alpha\beta}{}^\gamma \tau_\gamma\,,$$

- and invariant nondegenerate symmetric bilinear forms (ex. Killing forms),

$$\langle T_A, T_B \rangle_{\mathfrak{l}} = g_{AB}\,, \qquad \langle t_a, t_b \rangle_{\mathfrak{h}} = g_{ab}\,, \qquad \langle \tau_\alpha, \tau_\beta \rangle_{\mathfrak{g}} = g_{\alpha\beta}\,.$$

LIE 3-GROUPS

The main purpose of all this structure is to generalize the notion of parallel transport from curves to surfaces to volumes:

- Given a 4-dimensional manifold $\mathcal{M}$, define a 3-connection (α, β, γ) as a triple of 3-algebra-valued differential forms,

$$
\begin{array}{rcll}
\alpha & = & \alpha^{\alpha}{}_{\mu}(x)\, \tau_{\alpha}\, \mathbf{d}x^{\mu} & \in \Lambda^1(\mathcal{M}, \mathfrak{g})\,, \\
\beta & = & \frac{1}{2}\beta^{a}{}_{\mu\nu}(x)\, t_a\, \mathbf{d}x^{\mu} \wedge \mathbf{d}x^{\nu} & \in \Lambda^2(\mathcal{M}, \mathfrak{h})\,, \\
\gamma & = & \frac{1}{3!}\gamma^{A}{}_{\mu\nu\rho}(x)\, T_A\, \mathbf{d}x^{\mu} \wedge \mathbf{d}x^{\nu} \wedge \mathbf{d}x^{\rho} & \in \Lambda^3(\mathcal{M}, \mathfrak{l})\,.
\end{array}
$$

- Then introduce the line, surface and volume holonomies,

$$
g = \mathcal{P}\exp \int_{\mathcal{P}_1} \alpha\,, \qquad h = \mathcal{S}\exp \int_{\mathcal{S}_2} \beta\,, \qquad l = \mathcal{V}\exp \int_{\mathcal{V}_3} \gamma\,,
$$

- and corresponding curvature forms,

$$
\begin{array}{rcl}
\mathcal{F} & = & \mathbf{d}\alpha + \alpha \wedge \alpha - \partial\beta\,, \\
\mathcal{G} & = & \mathbf{d}\beta + \alpha \wedge^{\triangleright} \beta - \delta\gamma\,, \\
\mathcal{H} & = & \mathbf{d}\gamma + \alpha \wedge^{\triangleright} \gamma - \{\beta \wedge \beta\}\,.
\end{array}
$$

HIGHER GAUGE THEORIES

At this point one can construct the so-called $3BF$ theory, with the action:

$$S_{3BF} = \int_{\mathcal{M}} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}} \,.$$

- $3BF$ theory is a topological gauge theory,
- it is based on the 3-group structure,
- it is a generalization of an ordinary BF theory for a given Lie group G,

The physical interpretation of the Lagrange multipliers C and D:

- the $\mathfrak{h}$-valued 1-form C can be interpreted as the tetrad field, if $H = \mathbb{R}^4$ is the spacetime translation group:

$$C \to e = e^a{}_\mu(x)\, t_a \, \mathbf{d}x^\mu \,,$$

- the $\mathfrak{l}$-valued 0-form D can be interpreted as the set of real-valued matter fields, given some Lie group L:

$$D \to \phi = \phi^A(x)\, T_A \,.$$

HIGHER GAUGE THEORIES

How to choose the 3-group? The simplest example — the (trivial) Standard Model 3-group:

$$G = SO(3,1) \times SU(3) \times SU(2) \times U(1)\,, \qquad H = \mathbb{R}^4\,, \qquad L \text{ to be discussed.}$$

- boundary maps are trivial — for all $l \in L$ and $\vec{v} \in H$, we define

$$\delta l = 1_H = 0\,, \qquad \partial \vec{v} = 1_G\,,$$

- the bracket is trivial — for all $\vec{u}, \vec{v} \in H$, we define

$$\{\vec{u}, \vec{v}\} = 1_L\,,$$

- the action $\triangleright$ of G on itself is via the adjoint representation, the action on H is via vector representation for the $SO(3,1)$ sector and via trivial representation for the $SU(3) \times SU(2) \times U(1)$ sector.

- the action of G on L is nontrivial and depends on the choice of L (to be discussed).

One can verify that all axioms of a 3-group are satisfied.

HIGHER GAUGE THEORIES

How to choose L? Study the $3BF$ action:

$$S_{3BF} = \int_{\mathcal{M}} B^\alpha \wedge \mathcal{F}^\beta g_{\alpha\beta} + e^a \wedge \mathcal{G}^b g_{ab} + \phi^A \mathcal{H}^B g_{AB} \,.$$

- The indices α of G split according to its structure, as $\alpha = (ab\,,\, i)$, giving the connection and curvature

$$\alpha = \omega^{ab} J_{ab} + A^i \tau_i \,, \qquad \mathcal{F} = R^{ab} J_{ab} + F^i \tau_i \,.$$

- The vectorial action of $SO(3,1)$ on $H = \mathbb{R}^4$ implies the Minkowski signature of the bilinear invariant on $\mathbb{R}^4$, so that

$$g_{ab} = \eta_{ab} \equiv \mathrm{diag}(-1,+1,+1,+1)\,.$$

- Given that $\phi = \phi^A T_A$, we have one real-valued field $\phi^A(x)$ for each generator $T_A \in \mathfrak{l}$.

THE STANDARD MODEL

How many real-valued field components do we have in the matter sector of the Standard Model? The fermion sector gives us:

$$\left.\begin{matrix} \begin{pmatrix}\nu_e \\ e^-\end{pmatrix}_L & \begin{pmatrix}u_r \\ d_r\end{pmatrix}_L & \begin{pmatrix}u_g \\ d_g\end{pmatrix}_L & \begin{pmatrix}u_b \\ d_b\end{pmatrix}_L \\ & & & \\ (\nu_e)_R & (u_r)_R & (u_g)_R & (u_b)_R \\ & & & \\ (e^-)_R & (d_r)_R & (d_g)_R & (d_b)_R \end{matrix}\right\} = 16 \ \frac{\text{spinors}}{\text{family}} \times$$

$$\times 3 \text{ families } \times 4 \ \frac{\text{real-valued fields}}{\text{spinor}} \ = 192 \text{ real-valued fields } \phi^A \,.$$

The Higgs sector gives us:

$$\left.\begin{pmatrix}\phi^+ \\ \phi_0\end{pmatrix}\right\} = 2 \text{ complex scalar fields } = 4 \text{ real-valued fields } \phi^A \,.$$

This suggests the structure for L in the form:

$$L = L_{\text{fermion}} \times L_{\text{Higgs}} \,, \qquad \dim L_{\text{fermion}} = 192 \,, \qquad \dim L_{\text{Higgs}} = 4 \,.$$

THE STANDARD MODEL

The action $\triangleright : G \times L \to L$ specifies the transformation properties of each real-valued field ϕ^A with respect to Lorentz and internal symmetries.

- For example, in

$$\begin{pmatrix} u_b \\ d_b \end{pmatrix}_L$$

  the action

$$g \triangleright u_b \,, \qquad g \in SO(3,1) \times SU(3) \times SU(2) \times U(1) \,,$$

  encodes that u_b consists of 4 real-valued fields which transform as:

  – a left-handed spinor wrt. $SO(3,1)$,

  – as a “blue” component of the fundamental representation of $SU(3)$,

  – and as “isospin $+\frac{1}{2}$” of the left doublet wrt. $SU(2) \times U(1)$.

- Moreover, G acts in the same way across families, suggesting the structure

$$L_{\text{fermion}} = L_{\text{1st family}} \times L_{\text{2nd family}} \times L_{\text{3rd family}} \,, \qquad \dim L_{k\text{-th family}} = 64 \,.$$

THE STANDARD MODEL

A realisation that reproduces the Standard Model:

- Choice of the 3-group:

$$L = \mathbb{R}^4(\mathbb{C}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G})\,,$$

$$G = SO(3,1) \times SU(3) \times SU(2) \times U(1)\,, \qquad H = \mathbb{R}^4\,,$$

where $\mathbb{G}$ is the Grassmann algebra.

- The constrained $3BF$ action:

$$S = \int B_{\hat\alpha} \wedge \mathcal{F}^{\hat\alpha} + e_{\hat a} \wedge \mathcal{G}^{\hat a} + \phi_{\hat A} \wedge \mathcal{H}^{\hat A}$$

$$+ \left(B_{\hat\alpha} - C_{\hat\alpha}{}^{\hat\beta} M_{cd\hat\beta} e^c \wedge e^d\right) \wedge \lambda^{\hat\alpha} - \left(\gamma_{\hat A} - e^a \wedge e^b \wedge e^c C_{\hat A}{}^{\hat B} M_{abc\hat B}\right) \wedge \lambda^{\hat A} - 4\pi i\, l_p^2\, \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c \phi_{\hat A} T^{d\hat A}{}_{\hat B} \phi^{\hat B}$$

$$+\zeta^{ab}{}_{\hat\alpha} \wedge \left(M_{ab}{}^{\hat\alpha} \varepsilon^{cdef} e_c \wedge e_d \wedge e_e \wedge e_f - F^{\hat\alpha} \wedge e_c \wedge e_d\right) + \zeta^{ab}{}_{\hat A} \wedge \left(M_{abc}{}^{\hat A} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - F^{\hat A} \wedge e_a \wedge e_b\right)$$

$$-\varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \, \left(\Lambda + M_{\hat A\hat B} \phi^{\hat A} \phi^{\hat B} + Y_{\hat A\hat B\hat C} \phi^{\hat A} \phi^{\hat B} \phi^{\hat C} + L_{\hat A\hat B\hat C\hat D} \phi^{\hat A} \phi^{\hat B} \phi^{\hat C} \phi^{\hat D}\right)\,.$$

CONCLUSIONS

- Higher gauge theory represents a formalism where gravity, gauge fields, fermions and Higgs are treated on an equal footing.
- The underlying algebraic structure of a 3-group classifies all fundamental fields by specifying groups L, H, G and their maps $\delta, \partial, \triangleright, \{\ ,\ \}$.
- This structure has natural geometrical interpretation of parallel transport along a curve, a surface, and a volume.
- The gauge group L specifies the complete matter sector of the Standard Model if one chooses

$$L = \mathbb{R}^4(\mathbb{C}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G})\,.$$

- The action $\triangleright$ of G on L specifies the transformation properties of matter fields.
- Nontrivial choices of the 3-group structure may provide new avenues for research on unification of all fields.

THANK YOU!