CATEGORICAL GENERALIZATION OF SPINFOAM MODELS

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TOPICS

- **Recapitulation of spinfoam models**
- **Introduction to higher category theory**
- **Introduction to the spincube model**
- **Applications**

SPINFOAM MODELS

Construction of a spinfoam model:

(1) rewrite GR action using the Plebanski formalism:

$$S = \int B_{ab} \wedge R^{ab} + \phi^{abcd} B_{ab} \wedge B_{cd},$$

(2) quantize the BF sector by (a) triangulating the spacetime manifold, (b) defining the path integral

$$Z = \int \mathcal{D}\omega \int \mathcal{D}B \exp\left[i \sum_{\Delta} B_{\Delta} R_{\Delta} \right] = \ldots = \sum_{\Lambda} \prod_{\Delta} \mathcal{A}_{\Delta}(\Lambda_{\Delta}) \prod_{\sigma} \mathcal{A}_{\sigma}(\Lambda_{\sigma}),$$

where Λ are irreducible representations of $SO(3,1)$, while $\mathcal{A}_{\Delta}$ and $\mathcal{A}_{\sigma}$ are chosen such that Z is a topological invariant;

(3) enforce the "simplicity constraint" by projecting the representations from $SO(3,1)$ to $SU(2)$, and by redefining the vertex amplitude $\mathcal{A}_{\sigma}$

Thus get a non-topological path integral, with local degrees of freedom.

SPINFOAM MODELS

Main achievements:

- well-defined nonperturbative quantum theory of gravity,
- both kinematical and dynamical sectors under control,
- defines one possible UV-completion.

Main drawback:

- matter coupling is problematic.

The culprit: tetrads are not explicitly present in the action!

object	symbol	spinfoam labeling
vertex	v	
edge	ϵ	
triangle	Δ	$j \in \mathbb{N}_0/2$
tetrahedron	τ	$\iota \in \mathbb{N}_0/2$
4-simplex	σ	

SPINFOAM MODELS

Matter coupling of mass terms:

$$\left.\begin{array}{ll}
\bullet\ \text{Scalar field:} & \int d^4x\,\sqrt{-g}\,m^2\phi^2 \\
\bullet\ \text{Dirac field:} & \int d^4x\,\sqrt{-g}\,m\bar{\psi}\psi \\
\bullet\ \text{Vector field:} & \int d^4x\,\sqrt{-g}\,m^2 g^{\mu\nu}A_\mu A_\nu \\
\bullet\ \text{Yukawa interaction:} & \int d^4x\,\sqrt{-g}\,c_{\text{Yukawa}}\,\phi\,\bar{\psi}\psi \\
\bullet\ \text{Higgs mechanism:} & \int d^4x\,\sqrt{-g}\,(c\langle\phi\rangle_{\text{vev}})^2 g^{\mu\nu}W_\mu W_\nu \\
\bullet\ \text{Cosmological constant:} & \int d^4x\,\sqrt{-g}\,\Lambda
\end{array}\right\}\ \text{all proportional to } {}^{(4)}V_\sigma$$

Discretization always proportional to the simplex 4-volume!

SPINFOAM MODELS

Nonunqueness of the 4-volume:

$$A_{123} = 28\,, \quad A_{124} = 42\,, \quad A_{125} = 43\,, \quad A_{134} = 32\,, \quad A_{135} = 26\,,$$
$$A_{145} = 27\,, \quad A_{234} = 42\,, \quad A_{235} = 38\,, \quad A_{245} = 28\,, \quad A_{345} = 24\,.$$

N	l_{12}	l_{13}	l_{14}	l_{15}	l_{23}	l_{24}	l_{25}	l_{34}	l_{35}	l_{45}	$-^{(4)}V_\sigma^2$
1	9.188	6.866	9.647	15.263	8.572	15.298	9.790	10.301	9.907	7.216	96358.047
2	9.751	6.180	15.423	9.195	9.420	9.135	15.179	11.610	8.769	7.755	83095.409
3	9.841	6.254	15.930	10.998	9.213	9.267	9.294	11.839	15.816	6.433	138374.305
4	9.936	6.799	9.492	10.013	8.329	10.153	9.948	12.365	15.309	5.825	14626.743
5	9.995	6.191	10.364	9.425	9.251	9.278	10.576	11.682	8.615	18.949	29435.832
6	10.010	6.334	10.261	15.153	8.992	9.341	8.940	12.964	10.411	6.579	65535.329
7	10.012	6.292	10.173	15.024	9.063	9.409	8.896	11.887	10.365	6.576	43342.444
8	10.152	7.352	8.705	9.266	7.644	15.291	10.615	11.388	15.097	6.465	70393.956
9	10.362	18.623	9.328	10.066	9.212	9.937	9.522	10.528	9.378	18.492	428991.593
10	10.669	18.040	8.539	8.688	8.425	10.780	16.077	10.868	10.257	6.831	184052.701
11	10.809	6.742	9.996	15.572	15.987	8.991	8.363	10.156	10.273	7.101	121442.024
12	10.888	17.907	8.255	8.851	8.106	15.831	10.485	11.059	9.978	6.932	139620.742
13	11.257	5.519	16.580	10.493	10.168	8.263	8.653	13.213	9.466	7.365	7145.581
14	11.451	6.478	10.104	8.715	16.443	8.608	10.288	10.800	8.600	17.808	162923.501
15	11.518	6.032	15.321	9.311	15.853	7.551	9.551	11.737	8.979	7.573	67437.306
16	11.710	7.059	9.385	7.464	8.084	19.156	12.714	13.118	9.413	7.386	2106.96
17	11.875	19.723	8.718	9.650	8.671	9.798	9.092	12.076	10.768	6.333	26923.715
18	13.967	6.434	10.134	8.083	19.418	8.362	10.683	13.002	10.254	6.697	4065.232
19	14.069	7.238	8.847	9.425	8.957	9.618	9.214	11.595	15.150	6.327	68517.683
20	14.739	6.577	9.842	8.854	10.111	8.807	9.966	10.986	8.320	17.665	119422.745

HIGHER CATEGORIES

Category theory:

- a category is a structure with "objects" and "morphisms",
- a group is a category with only one object and invertible morphisms.

2-category theory:

- a 2-category is a structure with "objects", "morphisms" and "2-morphisms",
- a 2-group is a category with only one object and invertible morphisms and 2-morphisms.

Crossed module: $(G, H, \triangleright, \partial)$

- G and H are Lie groups,
- $\triangleright$ is an action of G on H ($\triangleright : G \times H \to H$),
- ∂ is a homomorphism of H on G ($\partial : H \to G$).

Theorem: every 2-group is isomorphic to a crossed module.

POINCARÉ 2-GROUP

Properties of the Poincaré 2-group:

- $(G, H, \triangleright, \partial)$, where:

  $$G = SO(3,1), \qquad H = \mathbb{R}^4, \qquad \triangleright : SO(3,1) \times \mathbb{R}^4 \to \mathbb{R}^4 \qquad \partial : \mathbb{R}^4 \to SO(3,1)$$

- Lorentz group has a connection 1-form ω, but the 2-Poincaré structure generates in addition a 2-form β, such that (ω, β) is called a 2-connection, and transforms as

  $$\omega \to g^{-1}\omega g + g^{-1} dg, \qquad \beta \to g^{-1} \triangleright \beta, \qquad (g : \mathcal{M}_4 \to SO(3,1))$$
  $$\omega \to \omega + \underbrace{\partial\eta}_{0}, \qquad \beta \to \beta + d\eta + \omega \wedge^{\triangleright} \eta + \underbrace{\eta \wedge \eta}_{0}, \qquad (\eta : \mathcal{M}_4 \to \mathbb{R}^4)$$

- one can introduce line holonomies and surface holonomies

  $$g_l(\omega) = \exp \int_l \omega, \qquad h_f(\beta) = \exp \int_f \beta,$$

- one can associate the $BFCG$ action to the Poincaré 2-group:

  $$S = \int B_{ab} \wedge R^{ab} + C_a \wedge G^a, \qquad (G^a = d\beta^a + \omega^a{}_b \wedge \beta^b).$$

THE $BFCG$ ACTION

Note that the Lagrange multiplier C^a

- is a 1-form,
- transforms as

$$C \to g^{-1} \triangleright C, \qquad C \to C \quad \text{wrt. } \eta \text{ transformations},$$

- has an equation of motion $\nabla C^a = 0$.

The multiplier C has exactly the same properties as the tetrad e!

Therefore,

$$\left.\begin{array}{ll} \bullet\ \textbf{identify:} & C^a \equiv e^a, \\ \bullet\ \textbf{rename:} & BFCG \to BFEG, \end{array}\right\} \boxed{\textbf{KEY STEP}}$$

and rewrite the action as

$$S = \int B_{ab} \wedge R^{ab} + e^a \wedge G_a .$$

CONSTRAINED $BFCG$ ACTION

The $BFCG$ action can be constrained to give GR:

$$S = \int \underbrace{B_{ab} \wedge R^{ab} + e^a \wedge G_a}_{\text{topological sector}} - \underbrace{\phi_{ab} \left(B^{ab} - \varepsilon^{abcd} e_a \wedge e_b \right)}_{\text{constraint}} .$$

Equations of motion:

$$\phi^{ab} = R^{ab}, \qquad B^{ab} = \varepsilon^{abcd} e_c \wedge e_d, \qquad \beta^a = 0,$$

$$\varepsilon_{abcd} R^{bc} \wedge e^d = 0, \qquad \nabla e^a = 0.$$

Key properties:

- equivalent to general relativity,
- similar in structure to the Plebanski action,
- based on the 2-gauge theory for the Poincaré 2-group,
- contains tetrad fields in the topological sector!

THE SPINCUBE MODEL

Construction of the spincube model:

(1) start from the constrained $BFCG$ action for GR:

$$S = \int \underbrace{B_{ab} \wedge R^{ab} + e^a \wedge G_a}_{\text{topological sector}} - \underbrace{\phi_{ab} \left(B^{ab} - \varepsilon^{abcd} e_a \wedge e_b \right)}_{\text{constraint}} ,$$

(2) quantize the $BFCG$ sector by (a) triangulating the spacetime manifold, (b) defining the path integral

$$Z = \int \mathcal{D}\omega \int \mathcal{D}B \int \mathcal{D}e \int \mathcal{D}\beta \exp \left[i \sum_{\Delta} B_\Delta R_\Delta + i \sum_{l} e_l G_l \right] = \ldots =$$

$$= \sum_{\Lambda} \prod_{\epsilon} \mathcal{A}_\epsilon(\Lambda_\epsilon) \prod_{\Delta} \mathcal{A}_\Delta(\Lambda_\Delta) \prod_{\sigma} \mathcal{A}_\sigma(\Lambda_\sigma),$$

where Λ are irreducible 2-representations of Poincaré 2-group, while $\mathcal{A}_\epsilon$, $\mathcal{A}_\Delta$ and $\mathcal{A}_\sigma$ are chosen such that Z is a topological invariant (a 2-TQFT),

THE SPINCUBE MODEL

The spincube quantization procedure:

(3) enforce the simplicity constraint $B^{ab} = \varepsilon^{abcd} e_c \wedge e_d$ by projecting representations Λ to a subset that satisfies the Heron formula for the area of a triangle,

$$|m_\Delta| l_p^2 = A(\Delta) \equiv \sqrt{s(s-l_1)(s-l_2)(s-l_3)}, \qquad s \equiv \frac{1}{2}(l_1 + l_2 + l_3),$$

and redefine the vertex amplitudes $\mathcal{A}_\epsilon$, $\mathcal{A}_\Delta$ and $\mathcal{A}_\sigma$ so that the theory is finite and has a correct classical limit.

Thus get a non-topological path integral, with local degrees of freedom.

Main achievement:

object	symbol	spinfoam labeling	spincube labeling
vertex	v		
edge	ϵ		$l \in \mathbb{R}_0^+$
triangle	Δ	$j \in \mathbb{N}_0/2$	$m \in \mathbb{Z}$
tetrahedron	τ	$\iota \in \mathbb{N}_0/2$	$M = 1$
4-simplex	σ		

SPINCUBE MODEL

Matter coupling is straightforward:

- the explicit presence of the tetrads enables us to couple any matter field,

$$S = \int B_{ab} \wedge R^{ab} + e^a \wedge G_a - \phi_{ab} \left(B^{ab} - \varepsilon^{abcd} e_a \wedge e_b \right) +$$

$$+ i\kappa \int \varepsilon_{abcd}\, e^a \wedge e^b \wedge e^c \wedge \bar{\psi} \left(\gamma^d \overset{\leftrightarrow}{d} + \{\omega, \gamma^d\} + \frac{im}{2}\, e^d \right) \psi -$$

$$- i\frac{3\kappa}{4} \int \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c\, \bar{\psi} \gamma_5 \gamma^d \psi, \qquad (\kappa = \frac{8}{3}\pi l_p),$$

- the 4-volume is a unique function of the edge-lengths,

$$^{(4)}V_\sigma^2 = -\frac{1}{9216} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & l_{12}^2 & l_{13}^2 & l_{14}^2 & l_{15}^2 \\ 1 & l_{12}^2 & 0 & l_{23}^2 & l_{24}^2 & l_{25}^2 \\ 1 & l_{13}^2 & l_{23}^2 & 0 & l_{34}^2 & l_{35}^2 \\ 1 & l_{14}^2 & l_{24}^2 & l_{34}^2 & 0 & l_{45}^2 \\ 1 & l_{15}^2 & l_{25}^2 & l_{35}^2 & l_{45}^2 & 0 \end{vmatrix} .$$

APPLICATIONS

Why is all this useful?

- study the spectrum of the cosmological constant,

$$\Lambda_{\text{eff}} = \frac{l_p^2}{2L_\mu^4} + \Lambda_{\text{bare}} + \frac{1}{l_p^2} f(m_i, c_i, \lambda_i, \ldots), \qquad L_\mu \gg \frac{l_p}{\sqrt[4]{|\Lambda_{\text{bare}} l_p^2|}},$$

- study the gravity-matter entanglement,

$$|\Psi\rangle = \alpha |g_1\rangle \otimes |\phi_1\rangle + \beta |g_2\rangle \otimes |\phi_2\rangle,$$

- study the grand-unified models $(G, H, \triangleright, \partial)$ where

$$\begin{array}{ll} H = \mathbb{R}^4, & G = SO(3,1) \times SU(3) \times SU(2) \times U(1), \\ H = \mathbb{R}^4, & G = SO(3,1) \times SU(5), \end{array}$$

- and all sorts of other interesting physics...

THANK YOU!