INTRODUCTION TO GENERAL RELATIVITY

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Abstract

We shall give an informal introduction to Einstein's theory of gravity. Starting from two fundamental principles, we shall emphasize the connection between physics and geometry in general, and gravitational force in particular. Implementing these principles will enable us to derive Einstein's field equations, and generalize to other fundamental forces as well. In order to better illustrate the theory, some of its consequences visible in everyday life will be discussed. The lecture is intended to be accessible to broad audience, and consequently does not require much mathematical background. However, it should also be quite interesting for those already familiar with basic differential geometry.

1 Introduction

The origin of General Relativity basically lies in two principles: principle of relativity and principle of equivalence. Intuitively, the first gives a "map" from physics to geometry, while the second gives an opposite map, from geometry to physics:

$$\text{physics} \quad \underset{\text{PE}}{\overset{\text{PR}}{\rightleftarrows}} \quad \text{geometry.}$$

The point of this lecture is to introduce the principles of relativity and equivalence, to explain this diagram, and to give the physical interpretation through some examples.

2 Principle of relativity

The principle of relativity essentially states:

> **The physical theory must be expressed in a way that is independent of any particular choice of a coordinate system.**

The most important bit in the above statement is the definition of a "coordinate system". Namely, this is the point at which physics gets related to some geometry. This geometry is determined by the structure of the physical theory we are dealing with. As an example, take Newtonian mechanics of a single free particle. The physical theory is defined by the famous three Newton's laws. They describe a point particle moving through 3-dimensional space, as time flows. The second law of motion can then be written as

$$m\frac{d^2\vec{r}}{dt^2} = 0.$$

The natural "geometry" that this law implicitly assumes is the manifold $\mathcal{M} = \mathbb{R}^3 \times \mathbb{R}$ in which the motion is defined. One may introduce coordinates $(\vec{r}, t)$ on this manifold, and ask whether the second

Newton's law upholds the principle of relativity, i.e. does the above equation change when we switch to some other coordinate system. The answer is no. Newton's law is invariant only with respect to the so-called "inertial" coordinate transformations, which form a Galilean symmetry group.

If one wishes to reformulate Newtonian theory so that it upholds the principle of relativity, one needs to "covariantize" the equations. Namely, we can rewrite the equation in the form

$$mu^\mu \nabla_\mu u^\nu = 0,$$

where

$$u^\mu \equiv \frac{dx^\mu}{d\lambda}, \qquad x^\mu = (t, \vec{r}),$$

are the velocity 4-vector and coordinates on the manifold $\mathcal{M}$, ∇_μ is the covariant derivative, and λ is the real parameter counting points on the world line of the particle. This form of Newton's second law is invariant with respect to any, *arbitrary* changes of coordinates. It is important to emphasize that this new form has the *same physical content* as the noncovariant formulation (i.e. it has the same set of solutions), but it now upholds the principle of relativity.

The main result of this "covariantization" procedure is the new language — we have reformulated Newton's law of motion in the language of *geometric quantities*. Essentially, we have rewritten it as a tensor equation. Tensors are objects that are by construction invariant to arbitrary coordinate transformations, and thus the theory written in terms of tensors is automatically invariant to those transformations.

On this trivial example we see how we can implement the principle of relativity in an arbitrary physical theory — rewrite the theory in a diffeomorphism-invariant way, or in other words, rewrite the theory in the language of *geometric quantities*. This is the realization of the map

$$\text{physics} \qquad \xrightarrow{\text{PR}} \qquad \text{geometry}.$$

Technically, we are mostly interested to implement this principle in classical field theory. This goes as follows. Essentially all relativistic classical field theories are formulated in Minkowski space via the action

$$S[\phi_r] = \int_{\mathbb{R}^4} d^4x \, \mathcal{L}(\phi_r, \partial_\mu \phi_r, \eta_{\mu\nu}),$$

where $\phi_r(x)$ are components of the field, while $\mathcal{L}$ is the Lagrangian — a function of these fields, their first derivatives and the Minkowski metric tensor $\eta_{\mu\nu} = \text{diag}(-1,1,1,1)$. Requirement that the variation of this action with respect to fields ϕ_r be zero leads to classical Euler-Lagrange equations of motion for the fields. The Lagrangian is typically formulated so that the action is invariant with respect to global Poincaré transformations, and perhaps some other internal symmetries. The natural geometry that can serve as a configuration space for this theory is a vector bundle $\mathcal{M} = \mathbb{R}^4 \times V_d$, where V_d is a d-dimensional vector space on which field components ϕ_r live (here $r = 1, \dots, d$). The structure group G of the corresponding principal bundle is the symmetry group of the action. For example, the structure group of the Standard Model is $P(4) \times SU(3) \times SU(2) \times U(1)$.

As it stands, the theory is not invariant with respect to arbitrary coordinate transformations, and does not uphold the principle of relativity[1]. In order to rewrite it in generally covariant way, we rewrite it in the language of geometric quantities, i.e. we require all quantities appearing in the action to be tensors over $\mathcal{M}$. The way to do this is to transform the action S from Minkowski coordinates to arbitrary curvilinear coordinates. In other words, we introduce a change of variables

$$x^\mu \to x^{\mu'} = f^{\mu'}(x^\nu)$$

[1]It is important not to confuse the principle of *special* relativity, which states that the physical theory must be invariant with respect to Poincaré coordinate transformations, and the principle of *general* relativity, which states that the physical theory must be invariant with respect to any coordinate transformations (Poincaré or otherwise). Throughout the text, we discuss only the latter, and assume that any relevant classical field theory already satisfies the former.

where functions $f^{\mu'}(x^\nu)$ have a nonvanishing Jacobian. This induces the transformation of the action functional that is typically called *minimal coupling*:

$$d^4x \to d^4x\sqrt{-\tilde{g}}, \qquad \partial_\mu \to \tilde{\nabla}_\mu, \qquad \eta_{\mu\nu} \to \tilde{g}_{\mu\nu}(x).$$

Here $\tilde{g}_{\mu\nu}(x)$ is the Minkowski metric tensor in curvilinear coordinates, $\tilde{g}$ is its determinant, and $\tilde{\nabla}_\mu$ is the covariant derivative, which acts on the fields ϕ_r as

$$\tilde{\nabla}_\mu \phi_r(x) = \partial_\mu \phi_r(x) + \tilde{A}^s{}_{r\mu}\phi_s(x).$$

The connection $\tilde{A}^s{}_{r\mu}(x)$ is the Levi-Civita connection for the vector bundle $\mathcal{M}$,

$$\tilde{A}^r{}_{s\mu}(x) = \tilde{A}^a{}_\mu(x)(T_a)^r{}_s,$$

where T_a are generators of G, and $(T_a)^r{}_s$ is their representation in the fiber V_d.

Note that in flat Minkowski coordinates the metric $\tilde{g}_{\mu\nu}(x)$ reduces to $\eta_{\mu\nu}$, while the connection $\tilde{A}^s{}_{r\mu}(x)$ reduces to zero. In other words, there exists a global coordinate system where metric and connection take these values in all spacetime points simultaneously.

One typically does not introduce coordinates or the metric for the fiber V_d, so if the vector bundle $\mathcal{M}$ has internal symmetries in addition to $P(4)$ there is no concept of coordinate transformations or the metric on the fiber. However, one does introduce the connection, in a way that is compatible with the action of the whole structure group G. This connection is either equal to zero everywhere, or can be transformed to zero by moving to a different section of the bundle $\mathcal{M}$ via the action of the group.

All in all, after the introduction of the minimal coupling into the action, it becomes

$$S[\phi_r] = \int_{\mathcal{M}} d^4x\sqrt{-\tilde{g}}\mathcal{L}(\phi_r, \tilde{\nabla}_\mu\phi_r, \tilde{g}_{\mu\nu}),$$

and has exactly the same set of solutions as the original one — while being reformulated in a diffeomorphism invariant way, via geometric quantities. The benefit of this reformulation in geometric language is that the theory now obeys the principle of relativity.

This is the practical implementation of the principle of relativity in classical field theory, and it represents a realization of the map from physics to geometry in the sense that all physical quantities entering the theory simultaneously become geometric quantities (i.e. tensor fields) on the Minkowski space. In the context of internal symmetry group, the described procedure is sometimes also called *symmetry localization*, mostly in high-energy physics community.

3 Principle of equivalence

In the previous section we described how to implement the principle of relativity into a given theory which is a priori not invariant with respect to general coordinate transformations on a given geometry. However, we stressed again and again that the new covariantized theory has exactly the same physical content as the original one, the difference being only in invariance properties of the quantities used to describe the theory. This means that the theory still does not describe any new phenomena like the gravitational interaction. In order to introduce gravity into the theory and couple it to matter fields, one must specify the details of this coupling. That is the content of the *equivalence principle*.

Historically, the principle of equivalence that was first formulated by Einstein, and subsequently rephrased in several variants, from "weak" to "strong" equivalence principle and beyond. Essentially, the equivalence principle states the following:

> **The physics in the presence of the gravitational field is locally indistinguishable form the physics in flat Minkowski space.**

The key word here is "locally", which translates to "in the infinitesimal neighborhood of any point in spacetime". In other words, all laws of physics take their special-relativistic form at every particular point. However, the transition from one point to the next may be nontrivial, and is the effect of the presence of gravitational field. As we shall see below, this amounts to introducing *curvature* into the spacetime manifold. The equivalence principle therefore forbids the possible situation in which the equations of motion[2] for matter fields with gravity would be explicitly different from analogous equations without gravity. In this way, the equivalence principle gives one precise prescription how to couple matter to gravity.

Another way to formulate the (weak) equivalence principle is to say that "gravitational mass is equal to inertial mass",

$$m_g = m_i.$$

Here gravitational mass m_g is the charge of the gravitational field, while the inertial mass m_i is the mass determined by the action of some force on the particle via second Newton's law of motion. This formulation is suitable for the application to particle mechanics, and provides a nice interpretation of the so-called "inertial forces". We shall revisit this later in one example.

How does one implement the statement of equivalence principle in practice, given a classical field theory discussed in the previous section? It is done in three steps. The first step is to implement the principle of relativity, which leads to the action

$$S[\phi_r] = \int_{\mathcal{M}} d^4x \sqrt{-\tilde{g}} \mathcal{L}(\phi_r, \tilde{\nabla}_\mu \phi_r, \tilde{g}_{\mu\nu}).$$

The second step is to generalize the theory by allowing the flat metric $\tilde{g}$ and flat connection $\tilde{A}$ to become *arbitrary*:

$$\tilde{g}_{\mu\nu}(x) \to g_{\mu\nu}(x), \qquad \tilde{A}^r{}_{s\mu}(x) \to A^r{}_{s\mu}(x).$$

This leads to the action

$$S[\phi_r, g_{\mu\nu}, A^r{}_{s\mu}] = \int_{\mathcal{M}} d^4x \sqrt{-g} \mathcal{L}(\phi_r, \nabla_\mu \phi_r, g_{\mu\nu}),$$

in which one no longer requires the metric and connection to be equal to $\eta_{\mu\nu}$ and zero globally, given some convenient coordinate system.

The third step is to add "kinetic" terms to the action, which will govern the dynamics of the metric and connection fields themselves. The explicit form of these kinetic terms is determined by the structure group G, and for concreteness we shall write the case given by the group

$$G = P(4) \times SU(3) \times SU(2) \times U(1)$$

which corresponds to the physically relevant case of the Standard Model[3]. The final action has the form:

$$S[\phi_r, g_{\mu\nu}, A^r{}_{s\mu}] = \int_{\mathcal{M}} d^4x \sqrt{-g} \mathcal{L}(\phi_r, \nabla_\mu \phi_r, g_{\mu\nu}) + \frac{1}{16\pi\gamma} \int_{\mathcal{M}} d^4x \sqrt{-g} R$$

$$-\frac{1}{4g_1} \int_{\mathcal{M}} d^4x \sqrt{-g} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4g_2} \int_{\mathcal{M}} d^4x \sqrt{-g} F^a_{\mu\nu} F^{\mu\nu}_a - \frac{1}{4g_3} \int_{\mathcal{M}} d^4x \sqrt{-g} F^k_{\mu\nu} F^{\mu\nu}_k.$$

Each kinetic term is an integral of some curvature scalar of the connection coming from the corresponding factor of the gauge group G, while γ, g_1, g_2, g_3 are called coupling constants.

Now we shall revisit and discuss the meaning of each of these three steps.

[2] It is assumed that equations of motion are differential and local.

[3] Strictly speaking, the Standard Model represents a quantum version of the classical theory we are discussing. Furthermore, it is always formulated in Minkowski spacetime (i.e. the spacetime curvature is required to vanish everywhere) because only for that special case the quantization procedure is well-defined. Quantization of the standard model which includes spacetime curvature is a (very hard) open problem.

The first step was already discussed in the previous section. But the second step is a crucial novelty, and contains the essence of the equivalence principle. Namely, the generalization from flat metric and flat connection to arbitrary ones represents the freedom that equivalence principle is giving us, while also implementing its restriction. This can be seen as follows. In the infinitesimal neighborhood of any particular point, one can choose a convenient coordinate system, in which the metric g and connection A take their flat-space values[4], η and 0. One can always orient the basis vectors at the point x so that the metric tensor $g_{\mu\nu}(x)$ is a diagonal matrix at x. Furthermore, one can always adjust the norm of these basis vectors so that the diagonal values of g are exactly those of η. Similarly, one can always choose a section[5] of the bundle $\mathcal{M}$ such that at point x we have $A = 0$. Therefore, at that particular point the equations of motion for the fields $\phi_r(x)$ reduce to their Minkowski form — a restriction prescribed by the equivalence principle.

On the other hand, note that in general this restriction is required *locally*, i.e. *only at one point* in spacetime, and not simultaneously for all points. In other words, we do not demand the existence of a single global coordinate system on the bundle $\mathcal{M}$ in which $g = \eta$ and $A = 0$ for all points at once. The fact that the above restriction is required only locally represents the freedom — again prescribed by the equivalence principle.

The third step is the natural consequence of the second step. This can be seen as follows. If the curvature of the connection of the structure group G is identically zero, then there is a theorem stating that there exist a global coordinate system such that $g = \eta$ and $A = 0$ everywhere. If that is the case, the generalization $\tilde{g} \to g$, $\tilde{A} \to A$ becomes trivial, which means that the equivalence principle is implemented in a trivial way. A nontrivial implementation therefore *requires* that the curvature be allowed to be nonzero in general. However, that means that scalars constructed from this curvature should be allowed to appear in the action. Which scalars should be chosen is in principle a matter of definition of the theory. However, in situations of physical interest we always require that the corresponding equations of motion for g and A be second-order partial differential equations, to have a well-defined Cauchy problem, etc. These requirements essentially limit the possibilities to a certain finite set of terms in the action, each entering with the appropriate coupling constant, which measures the relative strength of that type of interaction. The signs and values of these constants are determined experimentally.

At this point we need to step back and take a look at physical interpretation of the new action. It represents a theory in which the metric g and connection A are not just fixed quantities describing the geometry of the bundle $\mathcal{M}$, but rather dynamical fields over it, satisfying certain differential equations of motion. Furthermore, these equations of motion are coupled to equations of motion for the matter fields ϕ, so that the solution of these equations will depend not only on appropriate boundary conditions for g and A, but also on the values of matter fields ϕ. In other words, the geometry of the bundle $\mathcal{M}$ is not just an "arena", a "box" containing interacting matter fields. It becomes an *active participant in interactions* between matter fields — evolution of matter fields depends on the geometric properties of space, while geometric properties of space also evolve depending on the matter fields contained in that space. We see, therefore, that the equivalence principle is a realization of a map

$$\text{geometry} \qquad \xrightarrow{\text{PE}} \qquad \text{physics.}$$

4 Consequences

The principle of relativity and principle of equivalence provide one with a full theory describing space, time and matter. Both principles have deep consequences on the behavior of the theory. The principle of equivalence elevates the geometric quantities to the level of physical quantities, which have their own dynamical laws, i.e. equations of motion, and which interact with matter fields in a nontrivial way. Perhaps the most blunt way to see this is the *Einstein-Rosen bridge*, a specific solution of Einstein's

[4]It is always assumed that the metric g has the same signature as the Minkowski metric η.

[5]For the case of a completely general bundle $\mathcal{M}$ such a section may not exist. Therefore we restrict attention to those bundles for which this section exists. This essentially means that we exclude base manifolds which have nonzero torsion.

equations for the spacetime metric, in which one can see that the topology of space can change with time, based on the evolution of matter fields inside. Namely, one can consider two black holes whose interiors "connect" at certain moment, creating a handle in the topology of space, and thus changing its genus. After a period of (local) time, the handle is pinched and gets disconnected, reverting the geometry of space to its previous topology. That is an example where one sees the dynamical and physical character of geometry in its full glory. Another example would be the space which contains *gravitational waves*. Gravitational waves are also one of the solutions of the Einstein's equations for the spacetime metric, and they have a lot in common with, say, electromagnetic waves. They have two independent polarizations (i.e. degrees of freedom), they carry energy and momentum, and can interact with matter if it is present. This is also an illustrative example of the physical character of spacetime geometry.

The principle of relativity is also very powerful, and casts new light even on physics of everyday life. For an illustrative example, consider a passenger standing in the bus, which starts to brake rapidly at some moment in time. The passenger feels (and can measure) the "inertial force" pulling him towards the front of the bus. This force has quite an observable magnitude, as all of us have experienced when driving in a bus. The typical explanation in terms of Newtonian physics is that during the braking the rest-frame of the passenger is a non-inertial coordinate system, in which the second Newton's law has to be modified to account for this mysterious inertial force.

On the other hand, we know that in nature there are four fundamental interactions in physics — strong, weak, electromagnetic and gravitational. What is not clear is which of these interactions is responsible for the inertial force the bus passenger experiences. The strong and weak forces are short-ranged, and thus cannot generate such macroscopic-type force. Electromagnetic force can also be ruled out, since it requires objects to be electrically charged, while the passenger is not. This leaves us with the gravitational interaction, which exhibits some good properties such as proper dependence between the magnitude of the force and the mass of the passenger (i.e. equality of gravitational and inertial masses — equivalence principle). However, what is the *source* of this force? We know that the gravitational force is extremely weak, and in order to generate a horizontal inertial force comparable in magnitude with the vertical gravitational pull of the Earth, one needs a gravitational source of the size of a planet — in front of the bus! But there is no such source, as can be witnessed by a passenger standing on the sidewalk beside the moving bus. He feels no gravitational pull in the horizontal direction. How can these apparently contradictory observations be reconciled?

The answer lies in the principle of relativity. By its virtue, *all* coordinate frames are treated on equal footing, and the laws of physics must be formulated in a way that is independent on whether the passenger is in an "inertial" or "non-inertial" reference frame. Namely, the general-relativistic definition of "inertial" frame of reference is the "freely-falling" frame — the one in which no force other than gravitational acts on the observer. In that frame, a *local* observer will not feel even that gravitational force (equivalence principle!), and can write down the second Newton's law of motion. However, it is important to note that "freely-falling" means falling in some external gravitational field. In principle, it is the resultant gravitational field of all other material objects in the Universe. However, given that gravitational interaction decreases with distance, the only relevant object in whose gravitational field the Earth and its static observer on the sidewalk are freely-falling in is — the Sun. Namely, the Earth, the bus, and both observers are following a free-fall trajectory (a *geodesic*) around the Sun. Consequently, none of the observers should feel the Sun's gravitational field. However, when the bus starts to brake, the friction force (electromagnetic in nature) pushes the bus and its passenger off the free-fall trajectory. As a consequence, the passenger starts to feel the gravitational force of the Sun, equal in magnitude to the braking force that pushes him off the geodesic. Therefore, from the point of view of the passenger, the bus is in an "unusual" relative motion to the Sun, and has to take into account the Sun's gravitational field when he writes second Newton's law of mechanics. And that is precisely the "inertial" force that he feels. Conclusion — the inertial force is of gravitational origin, and its source is the Sun, which is massive enough to generate a force of macroscopic magnitude that the passenger feels.

These were some illustrative examples of principles of relativity and equivalence in action, which do not require a detailed mathematical analysis in order to be understood qualitatively. There are even more such examples, but we shall leave them for some other lecture. :-)