SCHWARZSCHILD SOLUTION IN GENERAL RELATIVITY

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Contents

Chapter 1

The meaning of the metric tensor

We begin with the definition of distance in Euclidean 2-dimensional space. Given two points A and B in the plane $\mathbb{R}^2$, we can introduce a Cartesian coordinate system and describe the two points with coordinates (x_A, y_A) and (x_B, y_B) respectively. Then we define the *distance* between these two points as:

$$s = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}.$$

Note:

- One can prove that the distance is *invariant* with respect of the choice of the coordinate system, provided that the new coordinate system is also orthogonal and has the same unit scale. (Below we shall generalize this statement to *all* coordinate systems.)
- One can naturally generalize this definition to the case of D-dimensional Euclidean space, $\mathbb{R}^D$, as:

  $$s^2 = \sum_{\mu=1}^{D} (x_A^\mu - x_B^\mu)^2,$$

  and even further to so-called *pseudo-Euclidean* spaces, like the Minkowski[1] spacetime M_4:

  $$s^2 = -(t_A - t_B)^2 + (x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2.$$

- If one draws a picture of the two points in a coordinate system, one can immediately see that the *key ingredient* in the above definition is the **theorem of Pythagora** — one can draw a right triangle, where the hypotenuse is connecting points A and B, while the two catheti are parallel to the two coordinate axes.

If points A and B lie on a curve $\mathcal{C}$, one may wish to calculate the length of the curve between the two points. In order to do so, one usually employs the *differential* definition of distance between the infinitesimally neighboring points x^μ and $x^\mu + dx^\mu$,

$$ds^2 = \sum_{\mu=1}^{D} (dx^\mu)^2 \tag{1.1}$$

and integrates it along the curve:

$$s = \int_{\mathcal{C}} ds.$$

[1] For the Minkowski case we use the "spacelike" metric convention, and we denote time as the "zero-coordinate", ie.

$$t = x^0, \qquad x = x^1, \qquad y = x^2, \qquad z = x^3.$$

The differential expression for the distance is called a *line element*, and is fundamental to calculating lengths of all geometric objects.

Suppose now that we want to introduce new coordinates, via a change of variables. In general, new coordinates are called *curvilinear coordinates*, and are defined as

$$x^{\mu'} = x^{\mu'}(x^\nu), \qquad \mu', \nu \in \{1, \ldots, D\}.$$

The only condition imposed on the functions $x^{\mu'}(x^\nu)$ is that they need to be *invertible*, meaning that we can go back to the original coordinates via

$$x^\nu = x^\nu(x^{\mu'}).$$

The necessary and sufficient condition for this to be true is that the Jacobian of the transformation must be nonzero[2]:

$$I = \det \left[\frac{\partial x^{\mu'}}{\partial x^\nu} \right] \neq 0.$$

A typical and most common example of curvilinear coordinates in three dimensions is the choice of *spherical coordinates*, r, θ, φ, introduced as:

$$x = r \sin\theta \cos\varphi, \qquad y = r \sin\theta \sin\varphi, \qquad z = r \cos\theta. \tag{1.2}$$

If one differentiates these expressions and substitutes them into the line element, one obtains the following nontrivial expression:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2.$$

We see that this is very peculiar, since the definition of distance appears to depend (in a nontrivial way) on the *choice of the point* where it is to be calculated. In general curvilinear coordinates one obtains a completely general expression for the line element:

$$\begin{aligned} ds^2 &= \sum_{\mu=1}^{D} \sum_{\nu=1}^{D} g_{\mu\nu}(x) dx^\mu dx^\nu \\ &= g_{\mu\nu}(x) dx^\mu dx^\nu \qquad \text{(by Einstein summation convention)} \\ &= g_{11}(x)(dx^1)^2 + g_{12}(x) dx^1 dx^2 + \cdots + g_{DD}(x)(dx^D)^2. \end{aligned}$$

The matrix with elements $g_{\mu\nu}(x)$ is called the **metric tensor**. Note:

- The metric tensor can always be assumed symmetric, because the differentials dx^μ and dx^ν commute.
- If one starts from the coordinates x^μ with metric $g_{\mu\nu}(x)$ and transforms to coordinates $x^{\mu'}$ with metric $g_{\mu'\nu'}(x')$, one can prove that the absolute value of the Jacobian of the transformation is:

$$|I| = \left| \det \left[\frac{\partial x^{\mu'}}{\partial x^\nu} \right] \right| = \frac{\sqrt{|g|}}{\sqrt{|g'|}},$$

where $g = \det g_{\mu\nu}$ and $g' = \det g_{\mu'\nu'}$.

[2]Except maybe for some particular choices of coordinates which are called "singular".

- As a consequence of the condition $I \neq 0$ and the above equation, we see that the metric is *always regular*,

$$\det g_{\mu\nu}(x) \neq 0.$$

This in turn means that the inverse matrix always exists, and is denoted $g^{\mu\nu}(x)$. The statement that these two tensors are inverse to each other can be conveniently written as

$$g_{\mu\nu} g^{\nu\lambda} = \delta^\lambda_\mu \equiv \begin{cases} 1 & \text{when } \lambda = \mu, \\ 0 & \text{when } \lambda \neq \mu. \end{cases}$$

The δ^μ_ν is called *the Kronecker symbol.*

- Since we started from the line element (1.1), there *always exists* a coordinate transformation from any curvilinear coordinates to the Cartesian, such that in Cartesian coordinates the metric has the form[3]:

$$[g_{\mu\nu}(x)] = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}.$$

At this point we make a **huge generalization** and **drop the final property**:

Define the distance via the line element

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$$

where $g_{\mu\nu}(x)$ is a **completely arbitrary** function of coordinates x^μ, satisfying only the condition $\det g_{\mu\nu} \neq 0$.

This definition of distance is the generalization of the **theorem of Pythagora** to **curved spaces**. Intuitively, one can understand this as follows. Dropping the above property means that, starting from some curvilinear coordinates, it might not be possible to transform to Cartesian coordinates. This means that all possible coordinates might be curvilinear, which in turn means that *there might be no straight lines* in such a space. Thus, such a space is called *curved.*

[3] In Minkowski spacetime it has the form

$$[g_{\mu\nu}(x)] = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix},$$

and is denoted $\eta_{\mu\nu}$.

Chapter 2

Einstein equations and Schwarzschild solution

The Einstein equations are usually written in the following form[1]:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi\, T_{\mu\nu}.$$

Note:

- The quantity $G_{\mu\nu}$ is called *the Einstein tensor*, while $T_{\mu\nu}$ is called *stress-energy tensor*.
- The *Ricci tensor* $R_{\mu\nu}$ and *scalar curvature* R are defined as:

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda{}_{\mu\nu} - \partial_\nu \Gamma^\lambda{}_{\lambda\mu} + \Gamma^\lambda{}_{\lambda\sigma} \Gamma^\sigma{}_{\mu\nu} - \Gamma^\lambda{}_{\sigma\nu} \Gamma^\sigma{}_{\lambda\mu}, \qquad R = g^{\mu\nu} R_{\mu\nu}. \tag{2.1}$$

  The Ricci tensor is symmetric.

- The *Christoffel symbol* $\Gamma^\lambda{}_{\mu\nu}$ is defined as:

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right). \tag{2.2}$$

  It is symmetric with respect to two lower indices.

All in all, we see that on the left-hand side of Einstein equations we have $G_{\mu\nu}$ which is a function of the metric, its first derivatives and its second derivatives. On the right-hand side we have $T_{\mu\nu}$ which represents matter. So essentially, Einstein equations represent a set of 10 second-order partial differential equations for the 10 components of the metric tensor, with stress-energy as a source term. Thus,

[1]We work in a natural system of units where speed of light and Newton's gravitational constant are defined to be equal to one,

$$c = 299\,792\,458\,\frac{\text{m}}{\text{s}} = 1, \qquad \gamma = 6.67428 \cdot 10^{-11}\,\frac{\text{m}^3}{\text{kg}\,\text{s}^2} = 1.$$

This means that we have redefined the second and kilogram in terms of a meter, as:

$$1\,\text{s} \equiv 299\,792\,458\,\text{m}, \qquad 1\,\text{kg} \equiv 7.43 \cdot 10^{-28}\,\text{m}.$$

If we wish to return from the natural system of units to the international SI metric system, we need to determine what are the SI units of a given physical quantity, and then multiply it with 1, suitably expressed in those units using the above formulas. The correspondence is unique. Also, as well as in the SI system, in the natural system of units and one can employ the technique of dimensional analysis to check one's calculations — the dimensions (in meters) of the left-hand side of any equation must always match the right-hand side.

Einstein equations determine the **geometry of spacetime** (by providing the definition of distance — theorem of Pythagora), based on the **matter content** in that spacetime. In addition, motion of matter is determined by this geometry.

The fact that the motion of matter is determined by properties of geometry is called *the equivalence principle*, and is built in the Einstein equations. While this is not obvious, we shall demonstrate it later in several examples.

Simply put, matter tells geometry how to curve, while geometry tells matter how to move. In this way, geometry ceases to be just an “arena” where “physics happens”, but rather becomes an *active participant* in physical processes. This participation of dynamics of geometry in physical processes is called **gravitational interaction**.

Now we will demonstrate all this in the simplest nontrivial case — the static spherically symmetric solution of Einstein equations, called *Schwarzschild geometry*. Begin from flat Minkowski spacetime with the line element

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2,$$

introduce spherical coordinates via the change of variables (1.2), and obtain:

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2.$$

This is still a flat-spacetime line element, just expressed in curvilinear coordinates. *Generalize this line-element* in such a way to allow for curvature, while preserving the requirements of geometry being *static* and *spherically symmetric*. Static means that the metric should not depend on time, while spherically symmetric means that it should not depend on angles θ and φ, other than through already present terms. One can then show that it is enough to consider the following generalization for the line element:

$$ds^2 = -e^{2F(r)}dt^2 + e^{2H(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2,$$

where $F(r)$ and $H(r)$ are two functions to be determined by Einstein equations. Note that we have chosen to write them in the exponent only because of later computational convenience.

So construct first the left-hand side of Einstein equations. Read the metric and inverse metric tensors from the line element:

$$[g_{\mu\nu}] = \begin{bmatrix} -e^{2F} & & & \\ & e^{2H} & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{bmatrix}, \qquad [g^{\mu\nu}] = \begin{bmatrix} -e^{-2F} & & & \\ & e^{-2H} & & \\ & & \frac{1}{r^2} & \\ & & & \frac{1}{r^2\sin^2\theta} \end{bmatrix}.$$

Use this to calculate the Christoffel symbols using (2.2). The nonzero symbols are:

$$\Gamma^t{}_{tr} = F', \qquad \Gamma^r{}_{tt} = F' e^{2F-2H}, \qquad \Gamma^r{}_{rr} = H', \qquad \Gamma^r{}_{\theta\theta} = -re^{-2H}, \qquad \Gamma^r{}_{\varphi\varphi} = -r\sin^2\theta e^{-2H},$$

$$\Gamma^\theta{}_{r\theta} = \frac{1}{r}, \qquad \Gamma^\theta{}_{\varphi\varphi} = -\sin\theta\cos\theta, \qquad \Gamma^\varphi{}_{r\varphi} = \frac{1}{r}, \qquad \Gamma^\varphi{}_{\theta\varphi} = \cot\theta.$$

Next use this to construct the Ricci tensor, using (2.1). The nonzero components are:

$$R_{tt} = e^{2F-2H}\left(F'' + (F')^2 - F'H' + \frac{2}{r}F'\right), \qquad R_{rr} = -\left(F'' + (F')^2 - F'H' - \frac{2}{r}H'\right),$$

$$R_{\theta\theta} = 1 - e^{-2H}\left(1 + rF' - rH'\right), \qquad R_{\varphi\varphi} = R_{\theta\theta}\sin^2\theta.$$

Next contract the Ricci tensor with the metric to obtain the Ricci scalar:

$$R = -2e^{-2H}\left[F'' + \left(F' + \frac{2}{r}\right)(F' - H') + \frac{1}{r^2}\left(1 - e^{2H}\right)\right].$$

Finally, put all this together to form the Einstein tensor:

$$G_{tt} = -\frac{1}{r^2} e^{2F-2H} \left(1 - 2rH' - e^{2H}\right), \qquad G_{rr} = \frac{1}{r^2} \left(1 + 2rF' - e^{2H}\right),$$

$$G_{\theta\theta} = r^2 e^{-2H} \left[F'' + \left(F' + \frac{1}{r} \right) (F' - H') \right], \qquad G_{\varphi\varphi} = G_{\theta\theta} \sin^2\theta.$$

Note that the G_{tt} component of the Einstein tensor can be rewritten in the form

$$G_{tt} = \frac{1}{r^2} e^{2F} \frac{d}{dr} \left[r \left(1 - e^{-2H}\right) \right],$$

which will be convenient for integration later on.

Next we concentrate on the right-hand side of the Einstein equation. We are interested in modeling the simplest possible stress-energy tensor, namely one that represents a static ball of radius R and density $\rho(r)$ with the center in $r = 0$. Remembering the general formula for the stress-energy tensor of a fluid element with density ρ, pressure p, and 4-velocity u^μ,

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu},$$

we wish to describe the static fluid ($u_r = u_\theta = u_\varphi = 0$). So the stress-energy obtains the form

$$T_{tt} = \rho u_t u_t + p \left(u_t u_t + g_{tt} \right), \qquad T_{rr} = p g_{rr}, \qquad T_{\theta\theta} = p g_{\theta\theta}, \qquad T_{\varphi\varphi} = p g_{\varphi\varphi},$$

while other components vanish. Next, the 4-velocity vector must be normalized, $u_\mu u_\nu g^{\mu\nu} = -1$, which means that $u_t u_t = -g_{tt} = e^{2F}$, so we have

$$T_{tt} = \rho e^{2F}, \qquad T_{rr} = p e^{2H}, \qquad T_{\theta\theta} = p r^2, \qquad T_{\varphi\varphi} = p r^2 \sin^2\theta.$$

The density and pressure of the fluid can depend only on r due to the spherical symmetry, and must be zero for $r > R$, ie. outside the ball.

Finally, after substituting all these results into Einstein equations, $G_{\mu\nu} = 8\pi T_{\mu\nu}$, we obtain three equations:

- the t-t equation:
$$\frac{1}{r^2} e^{2F} \frac{d}{dr} \left[r \left(1 - e^{-2H}\right) \right] = 8\pi e^{2F} \rho(r),$$
- the r-r equation:
$$\frac{1}{r^2} \left(1 + 2rF' - e^{2H}\right) = 8\pi e^{2H} p(r),$$
- the θ-θ and φ-φ equations (which are identical):
$$r^2 e^{-2H} \left[F'' + \left(F' + \frac{1}{r} \right) (F' - H') \right] = 8\pi r^2 p(r).$$

All other equations are vacuous, $0 = 0$, and provide no information.

First discuss the t-t equation. Straightforward integration gives

$$H(r) = -\frac{1}{2} \ln \left(1 - \frac{2m(r)}{r} \right), \qquad \text{where} \qquad m(r) \equiv 4\pi \int dr\, r^2 \rho(r).$$

Choosing the initial condition $m(0) = 0$, we can interpret $m(r)$ as the total mass inside radius r, since it is defined as an integral of mass density ρ over the volume of a ball of radius r.

Next discuss the r-r equation. Solve it for F to obtain

$$F(r) = \int dr\, \frac{m(r) + 4\pi r^3 p(r)}{r\,[r - 2m(r)]}.$$

For $r < R$, the integral is complicated and we shall not discuss it. However, for $r > R$ we have $m(r) = M$ (total mass of the ball) and $p(r) = 0$ (zero pressure in vacuum), so $F(r)$ can be easily integrated. The result is

$$F(r) = \frac{1}{2}\ln\left(1 - \frac{2M}{r}\right),$$

where the constant of integration has been chosen so that in the limit $r \to \infty$ the line element recovers its Minkowski form (far away from the ball spacetime should be flat).

Finally, discuss the θ-θ equation. Substitute all previous results and (after a tedious calculation) obtain the following result:

$$p'(r) + F'(r)\,(\rho(r) + p(r)) = 0.$$

This is a differential equation that determines the radial pressure distribution of matter within the ball. This distribution is such that the repulsive pressure balances attractive gravity everywhere, thereby maintaining static configuration of matter inside the ball.

In what follows, we shall not be interested in the geometry within the ball, but rather only the geometry outside (for $r > R$). In this case, the line element has the form:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1 - \dfrac{2M}{r}} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2.$$

This is the famous Schwarzschild solution of Einstein equations, and defines the so-called **Schwarzschild geometry**.

Chapter 3

Physical interpretation and consequences

3.1 Gravitational time dilatation

Consider an observer sitting at the surface of the Earth, say at coordinates $r_E = R$, $\theta_E = const$, $\varphi_E = const$. Assume that the observer carries a clock with him. He can construct the local Minkowski coordinates in the immediate neighborhood, described by the line element

$$ds^2 = -d\tau_E^2 + dx^2 + dy^2 + dz^2.$$

Since the observer is at rest in his own coordinates, we have $dx = dy = dz = 0$, and the coordinate τ_E measures the *proper time* of that observer, ie. it represents the readout of his clock. Given that the line element ds is *invariant*, it must be the same in both local Minkowski coordinates and in Schwarzschild coordinates, so (since $dr_E = d\theta_E = d\varphi_E = 0$) we obtain:

$$d\tau_E = dt\sqrt{1 - \frac{2M}{R}}.$$

Next consider another observer, sitting in geosynchronous orbit at height $h > 0$ above the ground, ie. at coordinates $r_O = R + h$, $\theta_O = const$, $\varphi_O = const$. He also carries a clock which measures his proper time,

$$ds^2 = -d\tau_O^2.$$

Equating this to the Schwarzschild line element, and remembering that $dr_O = d\theta_O = d\varphi_O = 0$, we obtain a similar relation:

$$d\tau_O = dt\sqrt{1 - \frac{2M}{R+h}}.$$

Now assume that both observers measure the time of some physical process that begins at coordinate time t and ends at coordinate time $t + dt$. Since the coordinate time is the same for both observers, using above formulas we obtain:

$$d\tau_E = d\tau_O\sqrt{\frac{1 - \frac{2M}{R}}{1 - \frac{2M}{R+h}}}.$$

Note that the expression under the square root is always *strictly less than one*, so we have

$$d\tau_{\text{Earth}} < d\tau_{\text{Orbit}},$$

ie. **the clocks on Earth tick slower than clocks in orbit!** This effect is called *gravitational time dilatation.*

The gravitational time dilatation effect is not only measurable on Earth, but it is downright *important* in precision measurements, for example in the Global Positioning System. The GPS system is the very first **real-world industrial application of general relativity**.

3.2 Gravitational redshift

Consider an atom at rest at the surface of the Sun, undergoing deexcitation and emitting a pulse of light in the process. In the atom's rest frame it takes time $\Delta\tau_S$ to emit one wavelength. So given this proper time between two equal phases of the wave, and given that velocity of light is equal to $c = 1$ in rest frame of the atom, the wavelength of the light pulse is given as

$$\lambda_S = c\Delta\tau_S \equiv \Delta\tau_S.$$

Let the wave go radially upwards into orbit, and consider an observer on Earth, at a distance D from the surface of the Sun. The observer uses his rest frame to measure the time between two equal phases of the incoming wave. Since velocity of light is again $c = 1$, he measures the wavelength as

$$\lambda_E = c\Delta\tau_E \equiv \Delta\tau_E.$$

But as we have shown in the previous section, the time interval $\Delta\tau_S$ on the surface of the Sun and the time interval $\Delta\tau_E$ in the Sun's orbit where Earth resides are related via

$$\Delta\tau_S = \Delta\tau_E\sqrt{\frac{1-\frac{2M}{R}}{1-\frac{2M}{R+D}}},$$

which translates into

$$\lambda_S = \lambda_E\sqrt{\frac{1-\frac{2M}{R}}{1-\frac{2M}{R+D}}}.$$

We see that the wavelength of the light pulse emitted from the surface of the Sun is always *strictly less* than the wavelength of the same pulse measured by the observer on Earth:

$$\lambda_S < \lambda_E,$$

ie. **the wavelength of the outgoing pulse is being shifted towards larger values**. This effect is called *gravitational redshift*. Note that the energy is inversely proportional to the wavelength, which means that the observed photon has *less* energy than it had when it was emitted. So the energy of the photon is not conserved, due to the interaction with the gravitational field. In the terms of Newtonian mechanics, the photon needs to climb up the gravitational "potential well", and loses energy in the process.

3.3 Newton's law of gravity

Consider the nonrelativistic motion of a test particle in Schwarzschild geometry. Nonrelativistic means that the particle's velocity can be considered much less than the speed of light, as measured relative to the origin of Schwarzschild coordinates. It also means that the source of the gravitational field is *weak*. Otherwise a nonrelativistic test particle would obtain relativistic velocity during its motion, due to the strong field that acts on it.

First, we rewrite the Schwarzschild line element in SI units (suitable for nonrelativistic analysis),

$$ds^2 = -\left(1-\frac{2M}{r}\frac{\gamma}{c^2}\right)c^2dt^2 + \frac{1}{1-\dfrac{2M}{r}\dfrac{\gamma}{c^2}}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2.$$

and *linearize in* M, in order to implement the weak field requirement:

$$ds^2 = -\left(1 - \frac{2M}{r}\frac{\gamma}{c^2}\right)c^2dt^2 + \left(1 + \frac{2M}{r}\frac{\gamma}{c^2}\right)dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2.$$

To get a proper feeling of the magnitude of this approximation, note that this is reasonably valid for the gravitational field of the Earth and other planets, but not for the field near the surface of the Sun, for example.

Second, implement the nonrelativistic velocity requirement by taking the limit $\vec{v}/c \to 0$. Note that

$$\frac{\vec{v}}{c} \to 0 \qquad \Leftrightarrow \qquad \frac{1}{c}\frac{dx^\mu}{ds} \to 0 \qquad \text{for} \qquad \mu = r, \theta, \varphi.$$

This essentially means that we are to discard the correction term which multiplies dr^2, so we obtain the following line element:

$$ds^2 = -\left(1 - \frac{2M}{r}\frac{\gamma}{c^2}\right)c^2dt^2 + dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2.$$

Now note that the *spatial part of the metric is Euclidean*, and that we can easily transform from spherical coordinates back to Cartesian ones:

$$ds^2 = -\left(1 - \frac{2M}{r}\frac{\gamma}{c^2}\right)c^2dt^2 + dx^2 + dy^2 + dz^2.$$

So we see that in nonrelativistic approximation **time is curved**, while **space is flat**.

In order to study this geometry further, we need the equations of motion for the test particle in curved space. Those equations can be derived from Einstein equations, but we omit the derivation here. Neglecting the gravitational field produced by the particle itself (the "test particle" approximation), the resulting trajectory is in general called a **geodesic line**, and is determined by the following differential equations:

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0.$$

The affine parameter τ counts the points of the trajectory. The nonzero Christoffel symbols calculated from the above line element read (remember that $r \equiv \sqrt{x^2 + y^2 + z^2}$):

$$\Gamma^t{}_{tx} = \frac{M}{r^3}\frac{\gamma}{c^2}x, \qquad \Gamma^t{}_{ty} = \frac{M}{r^3}\frac{\gamma}{c^2}y, \qquad \Gamma^t{}_{tz} = \frac{M}{r^3}\frac{\gamma}{c^2}z,$$

$$\Gamma^x{}_{tt} = \gamma\frac{M}{r^3}x, \qquad \Gamma^y{}_{tt} = \gamma\frac{M}{r^3}y, \qquad \Gamma^z{}_{tt} = \gamma\frac{M}{r^3}z.$$

Now writing down explicitly the four components $\lambda = t, x, y, z$ of the geodesic equation, we obtain:

$$t: \qquad \frac{d^2t}{d\tau^2} + 2\frac{M}{r^3}\frac{\gamma}{c^2}\frac{dt}{d\tau}\left(x\frac{dx}{d\tau} + y\frac{dy}{d\tau} + z\frac{dz}{d\tau}\right) = 0,$$

$$x: \qquad \frac{d^2x}{d\tau^2} + \gamma\frac{M}{r^3}x\left(\frac{dt}{d\tau}\right)^2 = 0,$$

$$y: \qquad \frac{d^2y}{d\tau^2} + \gamma\frac{M}{r^2}y\left(\frac{dt}{d\tau}\right)^2 = 0,$$

$$z: \qquad \frac{d^2z}{d\tau^2} + \gamma\frac{M}{r^2}z\left(\frac{dt}{d\tau}\right)^2 = 0.$$

Consider first the t equation. Due to the nonrelativistic velocity approximation, we can neglect all the terms in the parentheses. The remaining first term is easily integrated to:

$$t = \tau$$

using a suitable initial conditions (the equal scale and equal origin for t and τ). The remaining x, y and z equations can be multiplied with orthonormal Cartesian basis vectors $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$ respectively, and collected together to obtain

$$\frac{d^2\vec{r}}{dt^2} + \gamma\frac{M}{r^2}\vec{e}_r = 0.$$

(Note that $\vec{e}_r \equiv \vec{r}/r$ and $\vec{r} \equiv x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$.) Finally, if we multiply this with the mass m of the test particle, we can rewrite the result as

$$m\vec{a} = \vec{F}_g,$$

which is the familiar **Newton's second law of dynamics**, while the force term

$$\vec{F}_g = -\gamma\frac{mM}{r^2}\vec{e}_r$$

is the **Newton's law of gravitation** for a body with spherical symmetry and mass M.

3.4 Embedding diagram

In order to visually demonstrate that Schwarzschild spacetime is actually curved, it is instructive to represent it as a curved surface embedded in a bigger Euclidean space. Given that we cannot actually draw more than 3 Euclidean dimensions faithfully, the usual example of this kind of embedding deals with a 2-dimensional surface in Schwarzschild geometry — it is a surface defined by $t = const$ and $\theta = \pi/2$. It is a spatial slice through a coordinate origin, and has the line element

$$ds^2 = \frac{1}{1 - \frac{2m(r)}{r}}dr^2 + r^2 d\varphi^2. \tag{3.1}$$

Note that we have written $m(r)$ instead of M, in order to include region inside of the ball of matter.

This kind of geometry can be represented in 3-dimensional Euclidean space. It is convenient to use cylindrical curvilinear coordinates in this space, defined as

$$x = r\cos\varphi, \qquad y = r\sin\varphi,$$

which transform the Euclidean line element

$$ds^2 = dx^2 + dy^2 + dz^2$$

into the form

$$ds^2 = dr^2 + r^2 d\varphi^2 + dz^2.$$

The procedure of embedding now goes as follows — represent the Schwarzschild equatorial surface as a surface in Euclidean space, given via parametric equations

$$r = r, \qquad \varphi = \varphi, \qquad z = z(r),$$

where r and φ are both coordinates and parameters, while $z(r)$ is the "elevation function", which actually determines the shape of the surface. Due to axial symmetry, it does not depend on φ. Now construct the line element for this surface in Euclidean space,

$$ds^2 = \left[1 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + r^2 d\varphi^2,$$

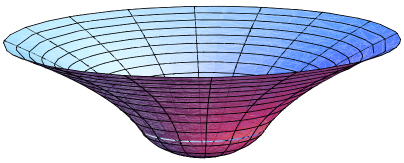
and require it to be identical to the line element of the equatorial plane in Schwarzschild geometry, (3.1), in order to represent equivalent internal geometries. Thereby obtain

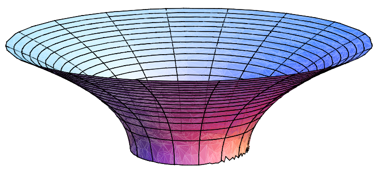
$$1 + \left(\frac{dz}{dr}\right)^2 = \frac{1}{1 - \frac{2m(r)}{r}},$$

which can be integrated to give $z(r)$:

$$z(r) = \int_0^r \frac{dr}{\sqrt{\frac{r}{2m(r)} - 1}}.$$

The function $m(r)$ is equal to M for $r > R$, while it depends on the detailed description of matter inside the ball, for $r < R$. Below are two pictures, one for some reasonable "homogeneous density" model of $m(r)$ (left picture), and the other for a black hole (right picture). Note that for the case of black hole the region $r < 2M$ cannot be embedded into Euclidean space, due to its Minkowski signature (we shall give a physical explanation for this the next chapter).





Also note that this 3-dimensional Euclidean embedding space is not physical, and exists only as an abstraction for the purposes of visualization of the surface. The only thing that physically exists is the surface itself, as part of the Schwarzschild geometry. Moreover, in Schwarzschild geometry this same surface looks like a *plane*, the equatorial plane, only with distorted definition of radial distances. This distortion manifests itself in Euclidean embedding as curvature of the surface.

Chapter 4

Black holes

The static configuration of the body of mass M and radius R is maintained due to the repulsive pressure of matter balancing the gravitational attractive force. While the gravitational force gets ever stronger with increasing M, the pressure within the body has an upper limit, based on the structure of particles that form the body, and the interactions between them. This means that for a large enough total mass M, the pressure will become insufficient to balance the gravitational attraction. The boundary value of M when this happens is called the *Chandrasekar limit*, and is roughly equal to three to five Solar masses.

If the mass of the body exceeds the Chandrasekar limit, there will be no equilibrium to maintain, and it starts contracting "all the way" into a ball of Planck radius ($R \simeq l_p \simeq 10^{-35}$ m). At those scales the effects of quantum gravity kick in, and Einstein's theory of gravity ceases to be valid. But outside that small sphere general relativity is an accurate description of spacetime geometry. The minuscule ball of matter is idealized as a single point with divergent energy density and spacetime curvature, and is called a *singularity*.

Thus, the geometry around the singularity has the Schwarzschild form all the way up to Planck scales, and such a configuration of matter and its surrounding geometry is called a **black hole**.

The above analysis suggests that we should consider the Schwarzschild line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1 - \dfrac{2M}{r}} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2.$$

for all possible values of the radial coordinate r greater than l_p, while keeping M large (the *strong field limit*). If we do so, we easily see that some very peculiar effects start happening around and inside the sphere of radius $r = 2M$. This sphere is called the **event horizon**. There are several important properties of the geometry inside and at the event horizon of a black hole.

4.1 Time stops at event horizon

Consider first the time dilatation equation derived above, rewritten in the form:

$$d\tau = d\tau_O \sqrt{\frac{1 - \frac{2M}{r}}{1 - \frac{2M}{r+h}}}.$$

Here τ is the proper time of the observer at rest at radius r from the center, while τ_O is the proper time of the observer at rest "in orbit" at radius $r + h$ from the center. If the first observer is at rest on the horizon, $r = 2M$, its **proper time stops**. Note, though, that this actually means that no observer can keep himself at rest there.

4.2 Finite proper time to fall through the horizon

Next consider the observer who starts from some radius $r_0 = 2M + h$ outside the event horizon, and starts falling radially towards the center. He carries with himself a clock that measures his proper time $d\tau$. Dividing the Schwarzschild line element by ds^2 and using the relation $ds^2 = -d\tau^2$, we obtain:

$$-1 = -\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{1 - \dfrac{2M}{r}}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2.$$

Since the observer is falling radially inwards, we have $\theta = const$ and $\varphi = const$, so the above equation reduces to:

$$-1 = -\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{1 - \dfrac{2M}{r}}\left(\frac{dr}{d\tau}\right)^2. \tag{4.1}$$

Next we need the time component of the geodesic equation for the observer, which in these coordinates has the form

$$\frac{d^2t}{d\tau^2} + \frac{2M}{r^2}\frac{1}{1 - \dfrac{2M}{r}}\frac{dt}{d\tau}\frac{dr}{d\tau} = 0.$$

Multiply this equation with $(1 - 2M/r)$ and integrate to find that

$$\frac{dt}{d\tau} = \frac{C}{1 - \dfrac{2M}{r}}, \tag{4.2}$$

where $C > 0$ is the constant of integration. We want this constant to be positive in order to ensure that both coordinate time t and proper time τ point "to the future". Now substitute this result back into the equation (4.1) to obtain

$$\frac{dr}{d\tau} = -\sqrt{C^2 - \left(1 - \frac{2M}{r}\right)}.$$

The choice of negative sign in front of the square root is because we consider the observers motion *towards* the black hole, so his radial coordinate should decrease with increasing time.

Finally, imagine that the observer is falling from the initial position r_0 all the way down to the event horizon $r = 2M$. Calculate his proper time of flight,

$$\Delta\tau = \int_{2M+h}^{2M} d\tau = \int_{2M}^{2M+h} \frac{dr}{\sqrt{C^2 - \left(1 - \frac{2M}{r}\right)}},$$

and, using (4.2), the coordinate time of flight

$$\Delta t = \int_{2M+h}^{2M} dt = \int_{2M}^{2M+h} \frac{C}{1 - \frac{2M}{r}} \frac{dr}{\sqrt{C^2 - \left(1 - \frac{2M}{r}\right)}}.$$

It is easy to see, by direct evaluation of the integrals, that the **proper time of flight is finite**, while the **coordinate time of flight is infinite**. Since the coordinate time is proportional to the proper time of the other observer (the one at rest in orbit at $r = 2M + h$), we conclude that the **orbit observer will never see the infalling observer reach the event horizon**, except asymptotically. However, the **infalling observer reaches the event horizon in finite proper time**, and moreover — if the appropriate integral is evaluated from $2M + h$ not down to $2M$ but even down to $r = 0$ — the **infalling observer will pass through event horizon and reach the center in finite proper time**.

4.3 Black holes are black

Consider an excited atom going through deexcitation and emitting a photon just before it passes through the event horizon. Imagine that the photon is emitted radially outwards, and detected by the stationary observer in orbit at radius $R = 2M + h$. The gravitational redshift that the photon undergoes is given by the formula

$$\lambda_{\text{detected}} = \lambda_{\text{emitted}}\sqrt{\frac{1-\frac{2M}{R}}{1-\frac{2M}{r}}},$$

where r is the radial coordinate of the atom at the time of deexcitation. If the atom is at the event horizon $r = 2M$, the **detected wavelength of the photon becomes infinite**. If the atom was just above the event horizon, the wavelength would be finite, but extremely large. But large wavelength means small frequency, which means small energy, which will escape detection if it is small enough. Therefore, **no light can be emitted or reflected from the event horizon**. The effect is the same also for all other forms of radiation and signals, since they all need to "climb up" an infinite gravitational potential well if they start from the horizon. Therefore, **since no radiation or signals can ever come out of the event horizon, black hole is indeed black**.

4.4 Black holes are holes

Once the observer has fallen through the event horizon, he might try to start his rocket engines in order to go back, or at least try to send a light signal to the observer outside the horizon. We have already seen that it would take an infinite amount of energy to "climb back up", so the observer is basically trapped inside the horizon, unable to get out.

By studying the orbits of test particles inside the event horizon (using geodesic equation) one can verify that it is impossible to climb up, no matter how much energy one has at one's disposal for, say, rocket boosters. In addition to the geodesic equations, there is an intuitive geometric argument why this is so.

Consider the observer in Minkowski space. He has the freedom to travel through space or to stand still. But he *does not have the same choice with respect to time*. Every observer invariably travels "to the future", without any ability to stop his motion through time. The reason for this lies in the **minus sign for time coordinate** in the Minkowski line element. Namely, we have

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2,$$

and if we divide by the proper time $d\tau = -ds$, we obtain:

$$-1 = -\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2 = \eta_{\mu\nu}u^\mu u^\nu.$$

From here we see that the magnitude of the 4-velocity u^μ of the observer is *constant*, and the 4-velocity vector is always *timelike* — ie. points to the future, regardless of the choice of coordinate system. Thus the observer can change his space velocity arbitrarily, but he *cannot have zero velocity through time*. The reason for this is that the coefficient in front of dt^2 in the line element is *negative*.

The same effect exists in curved spacetime. By looking at the Schwarzschild line element,

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\dfrac{2M}{r}}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2,$$

we see that the coefficient in front of dt^2 is negative (while other three are positive), and thus conclude that the observer must always have some nonzero velocity through time.

But this is only true for $r > 2M$! Once the observer falls through the event horizon, the **coefficients in front of dt^2 and dr^2 change signs!** This means that time coordinate and radial coordinate switch their roles, and consequently the observer can adjust his *orbital* velocity and velocity through *time*, but **cannot have zero radial velocity**. So in the same sense that the observer outside the horizon can choose not to travel through space, but invariably **must travel through time**, so the observer inside the horizon can choose not to travel through time and can have zero orbital velocity, but invariably **must fall toward the center**. The reason for this lies in the *geometry of spacetime*, and no amount of energy, motion or rocket boosters can prevent it from happening. Thus, once inside the horizon, there is no escape from reaching the singularity, or even maintaining a stable orbit around it. That is why **black holes are indeed holes**.

For further reading

[1] Lev D. Landau, Evgeny M. Lifshitz, “Course on Theoretical Physics Vol. 2: The Classical Theory of Fields”, Butterworth-Heinemann (1975), ISBN 978-0-750-62768-9.

[2] Charles W. Misner, Kip S. Thorne, John Archibald Wheeler, “Gravitation”, W. H. Freeman and Co. (1973), ISBN 978-0-7167-0344-0.